Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Industrial Engineering

Example 1

The open loop response, that is, the speed of the motor to a voltage input of 20V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w$$

If the initial speed is zero; use the Runge-Kutta 2nd order method and a step size of $h = 0.4\,\text{s}$ to find the speed at $t = 0.8\,\text{s}$.

Solution

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(w,t) = 1000 - 3w$$

Per Heun's method

$$w_{i+1} = w_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(w_i, t_i)$$

$$k_2 = f(w_i + h, t_i + k_1 h)$$

For $i = 0, t_0 = 0, w_0 = 0$

$$\begin{aligned}
k_1 &= f(t_0, w_0) \\
&= f(0,0) \\
&= 1000 - (3 \times 0) \\
&= 1000
\end{aligned}$$

$$\begin{aligned}
k_2 &= f(t_0 + h, w_0 + k_1 h) \\
&= f(0 + 0.4, 0 + (1000 \times 0.4)) \\
&= f(0.4, 400) \\
&= 1000 - (3 \times 400) \\
&= -200
\end{aligned}$$

$$\begin{aligned} w_1 &= w_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 0 + \left(\left(\frac{1}{2}(1000) + \frac{1}{2}(-200)\right) \times 0.4\right) \\ &= 0 + (500 - 100) \times 0.4 \\ &= 160 \text{ rad/s} \end{aligned}$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4\,\text{s}$$
$$w(0.4) \approx w_1 = 160\,\text{rad/s}$$

For $i = 1, t_1 = t_0 + h = 0 + 0.4 = 0.4, w_1 = 160$

$$\begin{aligned} k_1 &= f(t_1, w_1) \\ &= f(0.4, 160) \\ &= 1000 - (3 \times 160) \\ &= 520 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_1 + h, w_1 + k_1 h) \\ &= f(0.4 + 0.4, 160 + (520 \times 0.4)) \\ &= f(0.8, 368) \\ &= 1000 - (3 \times 368) \\ &= -104 \end{aligned}$$

$$\begin{aligned} w_2 &= w_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 160 + \left(\frac{1}{2}(520) + \frac{1}{2}(-104)\right) \times 0.4 \\ &= 160 + (208 \times 0.4) \\ &= 243.2 \text{ rad/s} \end{aligned}$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\,\text{s}$$
$$w(0.8) \approx w_2 = 243.2 \text{ rad/s}$$

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at $t = 0.8$ s is

$$w(0.8) = 303.09 \text{ rad/s}$$

The results from Heun's method are compared with exact results in Figure 1.

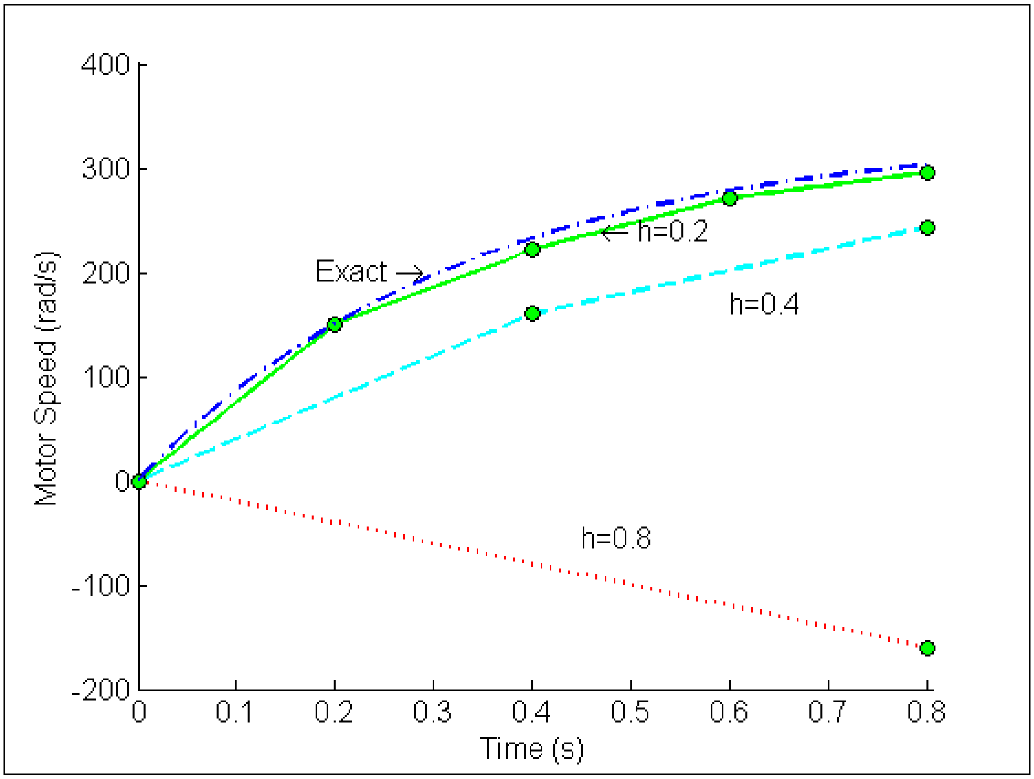

Figure 1 Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2 below.

Table 1 Effect of step size for Heun's method.

Step size, h	$w(0.8)$	E_t	$\lvert\epsilon_t\rvert\%$
0.8	-160.00	463.09	152.79
0.4	243.20	59.894	19.761
0.2	295.61	7.4823	2.4687
0.1	301.70	1.3929	0.45954
0.05	302.79	0.30613	0.10100

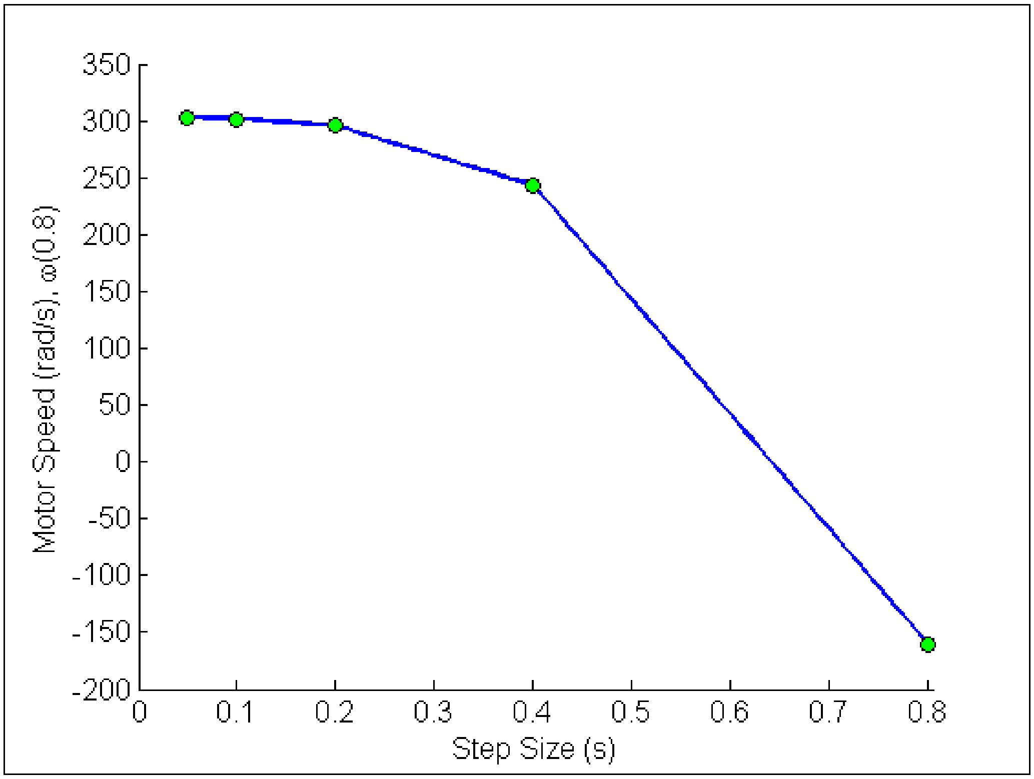

Figure 2 Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2nd order method results are shown as a function of step size.

Table 2 Comparison of Euler and the Runge-Kutta methods.

Step size, h	$w(0.8)$			
	Euler	Heun	Midpoint	Ralston
0.8	800	−160.00	−160.00	−160.00
0.4	320	243.20	243.20	243.20
0.2	324.8	295.61	295.61	295.61
0.1	314.11	301.70	301.70	301.70
0.05	308.58	302.79	302.79	302.79

In Figure 3, the comparison is shown over the range of time.

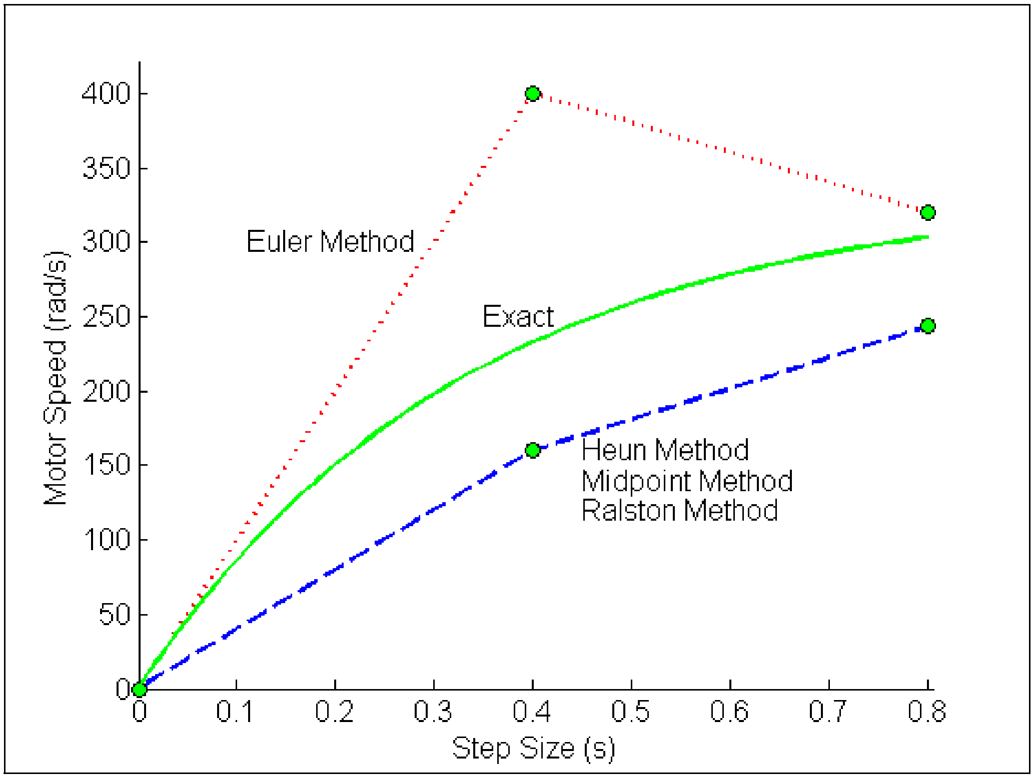

Figure 3 Comparison of Euler and Runge-Kutta methods with exact results over time ($h=0.4$).