Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Computer Engineering

Example 1

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150\ \mu\text{F}$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt}=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v(t)}{0.04},0\right)\right\}$$
$$v(0)=0$$

Using the Runge-Kutta 2nd order method, find the voltage across the capacitor at $t=0.00004\text{s}$. Use step size $h=0.00002\text{ s}$.

Solution

$$\frac{dv}{dt}=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$
$$f(t,v)=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

Per Heun's method

$$v_{i+1}=v_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$
$$k_1=f(t_i,v_i)$$
$$k_2=f(t_i+h,v_i+k_1h)$$

For $i=0, t_0=0, v_0=0$

$$k_1=f(t_0,v_0)$$
$$=f(0,0)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0))|-2-(0)}{0.04},0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(400,0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+400\}$$
$$= 2.6660\times10^{6}$$

$$k_2 = f(t_0+h, v_0+k_1h)$$
$$= f\left(0+0.00002, 0+\left(2.6660\times10^{6}\right)0.00002\right)$$
$$= f(0.00002, 53.32)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00002))|-2-(53.32)}{0.04},0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-933.01,0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$v_1 = v_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 0 + \left(\frac{1}{2}\left(2.6660\times10^{6}\right) + \frac{1}{2}(-666.67)\right)0.00002$$
$$= 0 + \left(1.3327\times10^{6}\right)0.00002$$
$$= 26.653\text{ V}$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002\text{ s}$$
$$v(0.00002) \approx v_1 = 26.653\text{ V}$$

For $i = 1, t_1 = t_0 + h = 0 + 0.00002 = 0.00002, v_1 = 26.653$

$$k_1 = f(t_1, v_1)$$
$$= f(0.00002, 26.653)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos((120\pi(0.00002))|-2-(26.653)}{0.04},0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-266.33,0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$k_2 = f(t_1+h, v_1+k_1h)$$

$$= f\left(0.00002 + 0.00002, 26.653 + (-666.67)0.00002\right)$$
$$= f\left(0.00004, 26.640\right)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos\left(120\pi\left(0.00004\right)\right)\right| - 2 - \left(26.640\right)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(-266.01, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + 0\right\}$$
$$= -666.67$$
$$v_2 = v_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 26.660 + \left(\frac{1}{2}\left(-666.67\right) + \frac{1}{2}\left(-666.67\right)\right)0.00002$$
$$= 26.660 + \left(-666.67\right)0.00002$$
$$= 26.647\ \text{V}$$

v_2 is the approximate voltage at

$$t = t_2 = t_1 + h = 0 + 0.00004 = 0.00004\ \text{s}$$
$$v\left(0.00004\right) \approx v_2 = 26.647\ \text{V}$$

The results from Heun’s method are compared with exact result of $v(0.00004) = 15.974\ \text{V}$ in Figure 1.

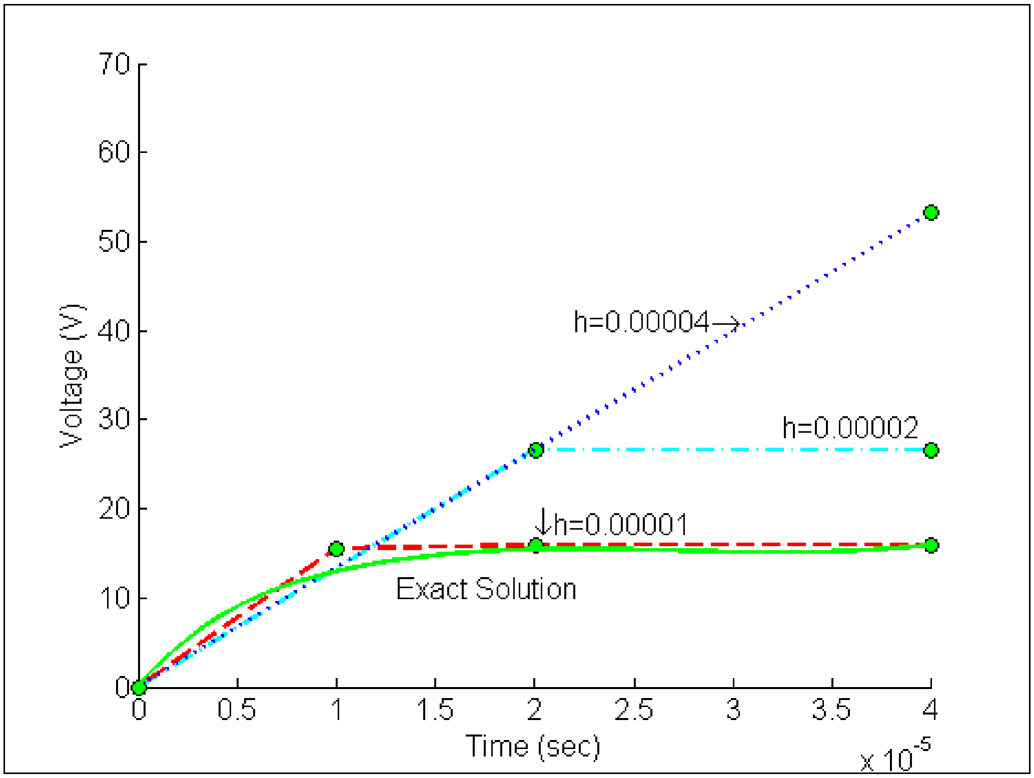

Figure 1 Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2 below.

Table 1 Effect of step size for Heun's method.

Step size, h	$v(0.00004)$	E_t	$\lvert \in_t \rvert \%$
0.00004	53.307	−37.333	233.71
0.00002	26.640	−10.666	65.771
0.00001	15.980	−0.0056605	0.035436
0.000005	15.918	0.055825	0.34947
0.0000025	15.970	0.0044682	0.027974

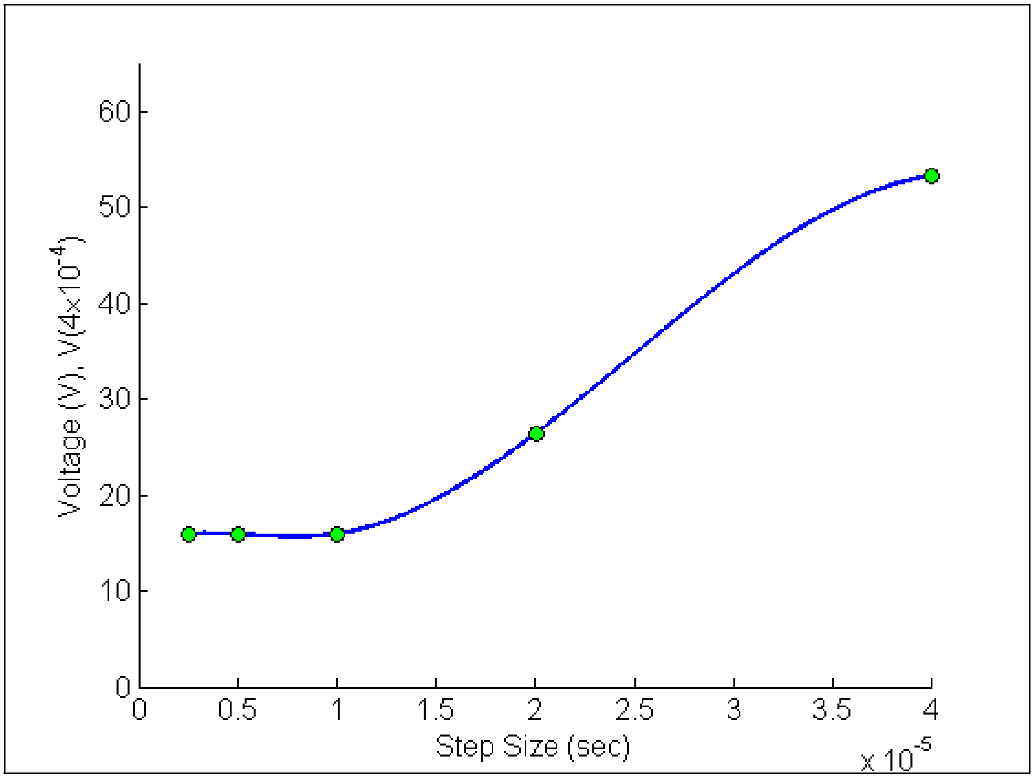

Figure 2 Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2nd order method results are shown as a function of step size.

Table 2 Comparison of Euler and the Runge-Kutta methods.

Step size, h	$v(0.00004)$			
	Euler	Heun	Midpoint	Ralston
0.00004	106.64	53.307	−0.026667	35.529
0.00002	53.307	26.640	−0.026667	17.751
0.00001	26.640	15.980	11.642	15.363
0.000005	15.996	15.918	15.917	15.917
0.0000025	15.993	15.970	15.968	15.968

While in Figure 3, the comparison is shown over the range of time.

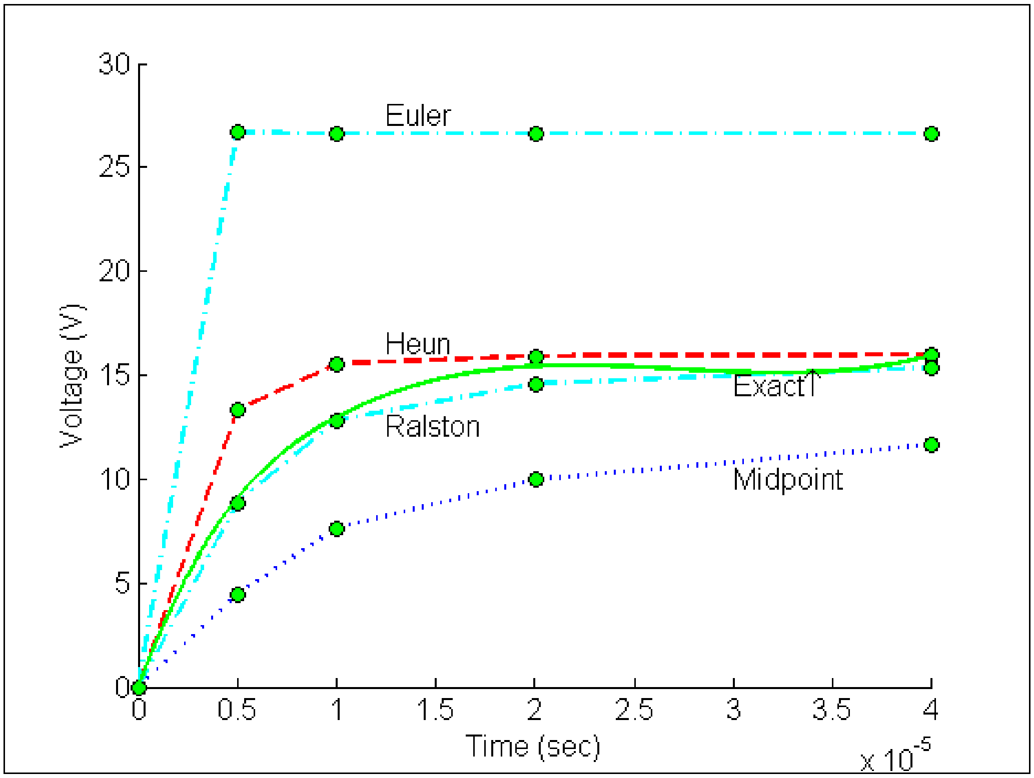

Figure 3 Comparison of Euler and Runge-Kutta methods with exact results over time.