Chapter 04.08
Gauss-Seidel Method – More Examples
Computer Engineering

Example 1

To infer the surface shape of an object from images taken of a surface from three different directions, one needs to solve the following set of equations.

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

The right hand side values are the light intensities from the middle of the images, while the coefficient matrix is dependent on the light source directions with respect to the camera. The unknowns are the incident intensities that will determine the shape of the object.

Find the values of x_1, x_2, and x_3 using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$x_1 = \frac{247 - 0x_2 - (-0.9701)x_3}{0.2425}$$

$$x_2 = \frac{248 - 0x_1 - (-0.9701)x_3}{0.2425}$$

$$x_3 = \frac{239 - (-0.2357)x_1 - (-0.2357)x_2}{-0.9428}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

we get

$$x_1 = \frac{247 - 0 \times 10 - (-0.9701) \times 10}{0.2425}$$
$$= 1058.6$$
$$x_2 = \frac{248 - 0 \times 1058.6 - (-0.9701) \times 10}{0.2425}$$
$$= 1062.7$$
$$x_3 = \frac{239 - (-0.2357) \times 1058.6 - (-0.2357) \times 1062.7}{-0.9428}$$
$$= -783.81$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{1058.6 - 10}{1058.6}\right| \times 100$$
$$= 99.055\%$$
$$|\in_a|_2 = \left|\frac{1062.7 - 10}{1062.7}\right| \times 100$$
$$= 99.059\%$$
$$|\in_a|_3 = \left|\frac{-783.81 - 10}{-783.81}\right| \times 100$$
$$= 101.28\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1058.6 \\ 1062.7 \\ -783.81 \end{bmatrix}$$

and the maximum absolute relative approximate error is 101.28%.

<u>Iteration #2</u>

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1058.6 \\ 1062.7 \\ -783.81 \end{bmatrix}$$

Now we get

$$x_1 = \frac{247 - 0 \times 1062.685 - (-0.9701) \times (-783.8116)}{0.2425}$$
$$= -2117.0$$
$$x_2 = \frac{248 - 0 \times (-2117.0) - (-0.9701) \times (-783.81)}{0.2425}$$
$$= -2112.9$$
$$x_3 = \frac{239 - (-0.2357) \times (-2117.0) - (-0.2357) \times (-2112.9)}{-0.9428}$$
$$= 803.98$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{(-2117.0) - 1058.6}{-2117.0}\right| \times 100$$

$$= 150.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{(-2112.9) - 1062.7}{-2112.9}\right| \times 100$$

$$= 150.30\%$$

$$\left|\in_a\right|_3 = \left|\frac{803.98 - (-783.81)}{803.98}\right| \times 100$$

$$= 197.49\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2117.0 \\ -2112.9 \\ 803.98 \end{bmatrix}$$

and the maximum absolute relative approximate error is 197.49%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	1058.6	99.055	1062.7	99.059	–783.81	101.28
2	–2117.0	150.00	–2112.9	150.295	803.98	197.49
3	4234.8	149.99	4238.9	149.85	–2371.9	133.90
4	–8470.1	150.00	–8466.0	150.07	3980.5	159.59
5	16942	149.99	16946	149.96	–8725.7	145.62
6	–33888	150.00	–33884	150.01	16689	152.28

After six iterations, the absolute relative approximate errors are not decreasing. In fact, conducting more iterations reveals that the absolute relative approximate error does not approach zero but approaches 149.99%.

SIMULTANEOUS LINEAR EQUATIONS

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