Introduction to Partial Differential Equations

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Transforming Numerical Methods Education for STEM Undergraduates

What is a Partial Differential Equation ?

- **Ordinary Differential Equations have only one independent variable**

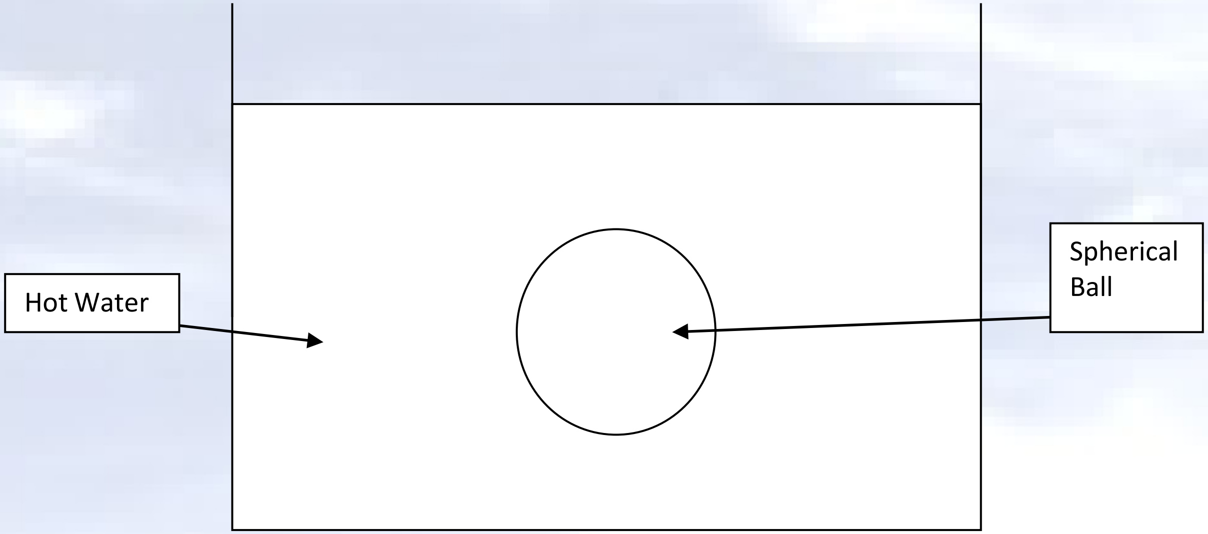
$$3\frac{dy}{dx}+5y^2=3e^{-x} \quad ,y(0)=5$$

- **Partial Differential Equations have more than one independent variable**

$$3\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=x^2+y^2$$

- **subject to certain conditions: where is the dependent variable, and x and y are the independent variables.**

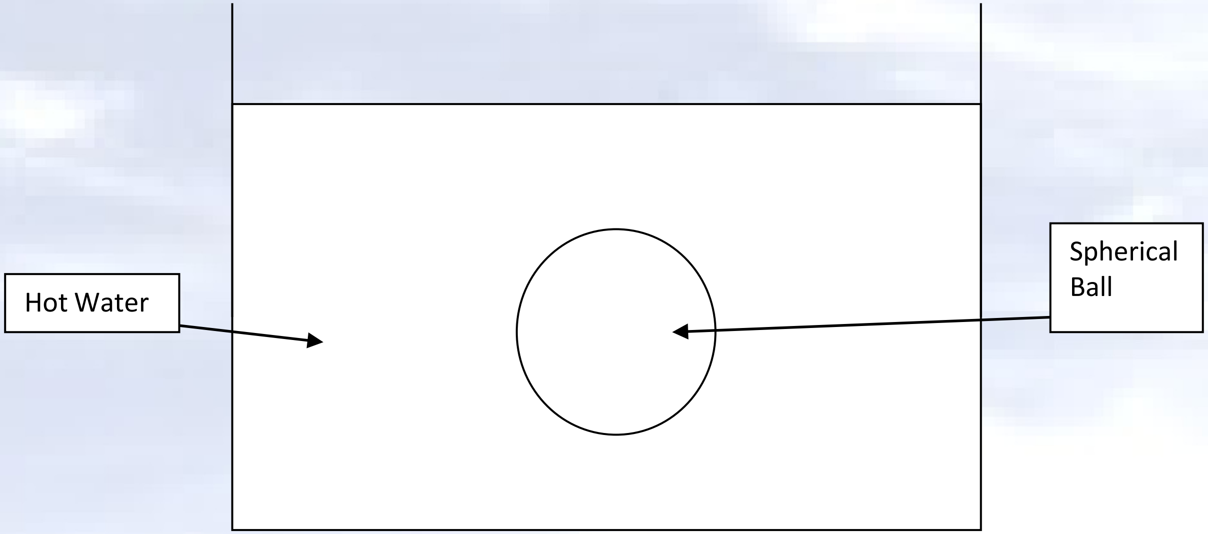
Example of an Ordinary Differential Equation



$$hA(\theta - \theta_a) = mC\frac{d\theta}{dt}$$

- **Assumption: Ball is a lumped system.**
- **Number of Independent variables: One (t)**

Example of an Partial Differential Equation



$$\frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{k}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}=\rho C\frac{\partial T}{\partial t}\quad ,t\geq 0,\ T(r,\theta,\phi,0)=T_a$$

- **Assumption: Ball is not a lumped system.**
- **Number of Independent variables: Four (r, θ,φ,t)**

Classification of 2nd Order Linear PDE's

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, and C are functions of x and y, and D is a function of

$$x, y, u \text{ and } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}.$$

Classification of 2nd Order Linear PDE's

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Can Be:

- Elliptic
- Parabolic
- Hyperbolic

Classification of 2nd Order Linear PDE's: Elliptic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

If $B^2 - 4AC < 0$, then equation is elliptic.

Classification of 2nd Order Linear PDE's: Elliptic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Example: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

where, $A = 1, B = 0, C = 1$ giving

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

therefore the equation is elliptic.

Classification of 2nd Order Linear PDE's: Parabolic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

If $B^2 - 4AC = 0$, then the equation is parabolic.

Classification of 2nd Order Linear PDE's: Parabolic

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

Example: $\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$

where, $A=k, B=0, C=0, D=-1$ giving

$$B^2-4AC \;=0-4(0)(k)\;=0$$

therefore the equation is parabolic.

Classification of 2nd Order Linear PDE's: Hyperbolic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

If $B^2 - 4AC > 0$, then the equation is hyperbolic.

Classification of 2nd Order Linear PDE's: Hyperbolic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Example: $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$

where, $A = 1, B = 0, C = -\frac{1}{c^2}$ giving

$$B^2 - 4AC = 0 - 4(1)(\frac{-1}{c^2}) = \frac{4}{c^2} > 0$$

therefore the equation is hyperbolic.

THE END