

Chapter 09.01

Golden Section Search Method

After reading this chapter, you should be able to:

1. *Understand the fundamentals of the Equal Interval Search method*
2. *Understand how the Golden Section Search method works*
3. *Learn about the Golden Ratio*
4. *Solve one-dimensional optimization problems using the Golden Section Search method*

Equal Interval Search Method

One of the simplest methods of finding the local maximum or local minimum is the Equal Interval Search method. Let's restrict our discussion to finding the local maximum of $f(x)$ where the interval in which the local maximum occurs is $[a, b]$. As shown in Figure 1, let's choose an interval of ε over which we assume the maximum occurs. Then we can compute $f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right)$ and $f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right)$. If $f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right) \geq f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right)$, then the interval in which the maximum occurs is $\left[\frac{a+b}{2} - \frac{\varepsilon}{2}, b\right]$, otherwise it occurs in $\left[a, \frac{a+b}{2} + \frac{\varepsilon}{2}\right]$. This reduces the interval in which the local maximum occurs. This procedure can be repeated until the interval is reduced to the level of our choice.

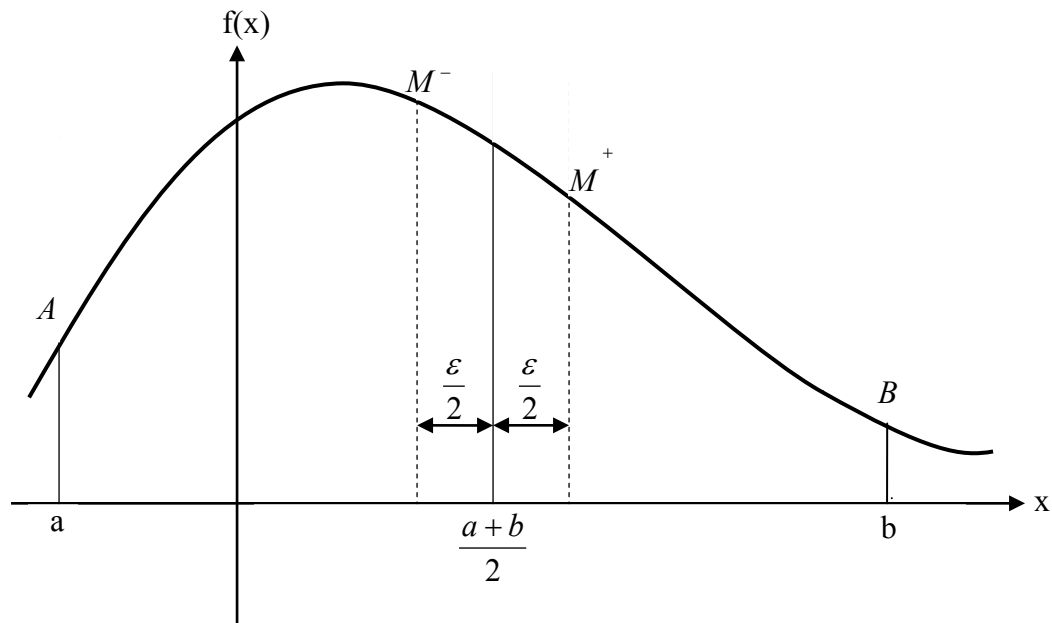


Figure 1A Equal interval search method (new **upper bound** can be identified).

Remarks:

As can be seen from the marked data points A, M^- , M^+ , and B on Figure 1A, the function values have increased from point A to point M^- , but then have decreased from point M^- to point M^+ . Whenever there is a sudden change in the pattern, such as from increasing the function value to decreasing its value, as shown in Figure 1A (or vice versa, as shown in Figure 1B, where $f_L < f_2 < f_1$ and then $f_1 > f_u$), then the new lower and upper bound bracket values can be found. In this case, the new lower bound remains to be the same as its previous lower bound (at point A), and the new upper bound can be found (at point M^+), as shown in Figure 1A.

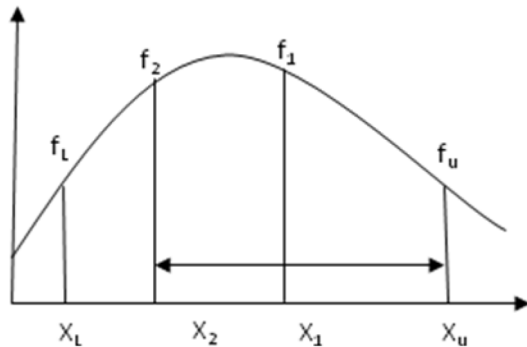


Figure 1B Equal interval search method (new **lower bound** can be identified).

Example 1

Consider Figure 2 below. The cross-sectional area A of a gutter with equal base and edge length of 2 is given by

$$A = 4 \sin \theta (1 + \cos \theta)$$

Using an initial interval of $[0, \pi/2]$, find the interval after 3 iterations. Use an initial interval $\varepsilon = 0.2$.

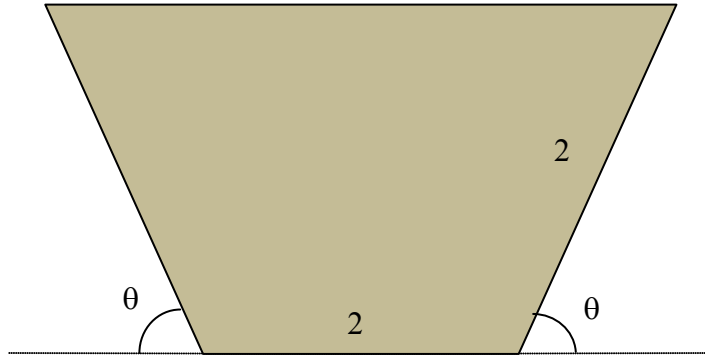


Figure 2 Cross section of the gutter.

Solution

If we assume the initial interval to be $[0, \pi/2] \cong [0, 1.5708]$ and choose $\varepsilon = 0.2$, then

$$\begin{aligned} f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right) &= f\left(\frac{0+1.5708}{2} + \frac{0.2}{2}\right) \\ &= f(0.88540) \\ &= 5.0568 \end{aligned}$$

$$\begin{aligned} f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right) &= f\left(\frac{0+1.5708}{2} - \frac{0.2}{2}\right) \\ &= f(0.6854) \\ &= 4.4921 \end{aligned}$$

Since $f(0.88540) > f(0.68540)$, the interval in which the local maximum occurs is $[0.68540, 1.5708]$.

Now

$$\begin{aligned} f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right) &= f\left(\frac{0.68540+1.5708}{2} + \frac{0.2}{2}\right) \\ &= f(1.2281) \\ &= 5.0334 \end{aligned}$$

$$\begin{aligned}
 f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right) &= f\left(\frac{0.68540 + 1.5708}{2} - \frac{0.2}{2}\right) \\
 &= f(1.0281) \\
 &= 5.1942
 \end{aligned}$$

Since $f(1.2281) < f(1.0281)$, the interval in which the local maximum occurs is $[0.68540, 1.2281]$.

Now

$$\begin{aligned}
 f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right) &= f\left(\frac{0.68540 + 1.2281}{2} + \frac{0.2}{2}\right) \\
 &= f(1.0567) \\
 &= 5.1957 \\
 f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right) &= f\left(\frac{0.68540 + 1.2281}{2} - \frac{0.2}{2}\right) \\
 &= f(0.8567) \\
 &= 5.0025
 \end{aligned}$$

Since $f(1.0567) > f(0.8567)$, then the interval in which the local maximum occurs is $(0.8567, 1.2281)$. After sixteen iterations, the interval is reduced to 0.02 and the approximation of the maximum area is 5.1961 at an angle of 60.06 degrees.

The exact answer is $\theta = 1.0472$ for which $f(\theta) = 5.1962$.

What is the Golden Section Search method used for and how does it work?

The Golden Section Search method is used to find the maximum or minimum of a unimodal function. (*A unimodal function contains only one minimum or maximum on the interval $[a, b]$.*) To make the discussion of the method simpler, let us assume that we are trying to find the maximum of a function. The previously introduced Equal Interval Search method is somewhat inefficient because if the interval is a small number it can take a long time to find the maximum of a function. To improve this efficiency, the Golden Section Search method is suggested.

As shown in Figure 3, choose three points x_l , x_1 and x_u ($x_l < x_1 < x_u$) along the x -axis with corresponding values of the function $f(x_l)$, $f(x_1)$, and $f(x_u)$, respectively. Since $f(x_1) > f(x_l)$ and $f(x_1) > f(x_u)$, the maximum must lie between x_l and x_u . Now a fourth point denoted by x_2 is chosen to be between the larger of the two intervals of $[x_l, x_1]$ and $[x_1, x_u]$. Assuming that the interval $[x_l, x_1]$ is larger than $[x_1, x_u]$, we would choose $[x_l, x_1]$ as the interval in which x_2 is chosen. If $f(x_2) > f(x_1)$ then the new three points would be

$x_l < x_2 < x_1$; else if $f(x_2) < f(x_1)$ then the new three points are $x_2 < x_1 < x_u$. This process is continued until the distance between the outer points is sufficiently small.

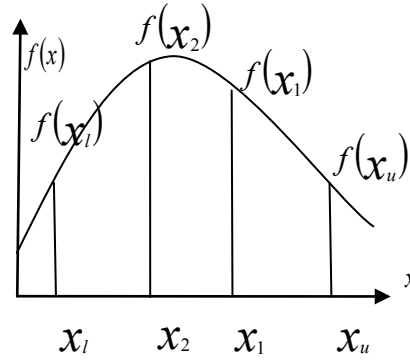


Figure 3 Cross section of the gutter.

How are the intermediate points in the Golden Section Search determined?

We chose the first intermediate point x_l to equalize the ratio of the lengths as shown in Eq. (1) where a and b are distance as shown in Figure 4. Note that $a + b$ is equal to the distance between the lower and upper boundary points x_l and x_u .

$$\frac{a}{a+b} = \frac{b}{a} \quad (1)$$

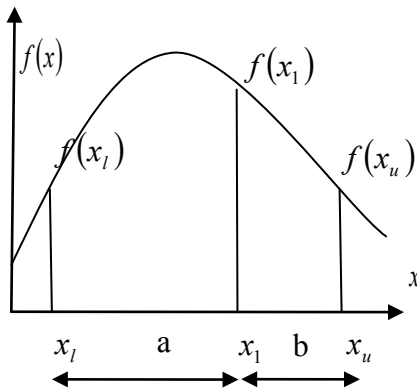


Figure 4 Determining the first intermediate point

The second intermediate point x_2 is chosen similarly in the interval a to satisfy the following ratio in Eq. (2) where the distances of a and b are shown in Figure 5.

$$\frac{b}{a} = \frac{a-b}{b} \quad (2)$$

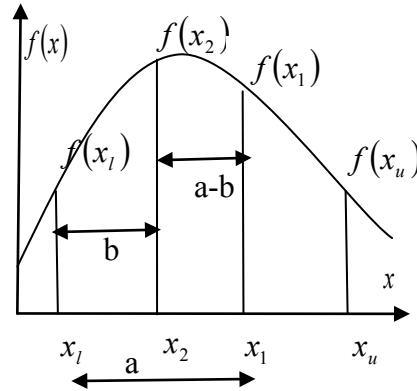


Figure 5 Determining the second intermediate point

Does the Golden Section Search have anything to do with the Golden Ratio?

The ratios in Equations (1) and (2) are equal and have a special value known as the Golden Ratio. The Golden Ratio has been used since ancient times in various fields such as architecture, design, art and engineering. To determine the value of the Golden Ratio let $R = a/b$, then Eq. (1) can be written as

$$1 + R = \frac{1}{R}$$

or

$$R^2 + R - 1 = 0$$

(3)

Using the quadratic formula, the positive root of Eq. (3) is

$$R = \frac{-1 + \sqrt{1 - 4(-1)}}{2}$$

$$= \frac{\sqrt{5} - 1}{2}$$

$$= 0.61803$$

(4)

In other words, the intermediate points x_1 and x_2 are chosen such that, the ratio of the distance from these points to the boundaries of the search region is equal to the golden ratio as shown in Figure 6.

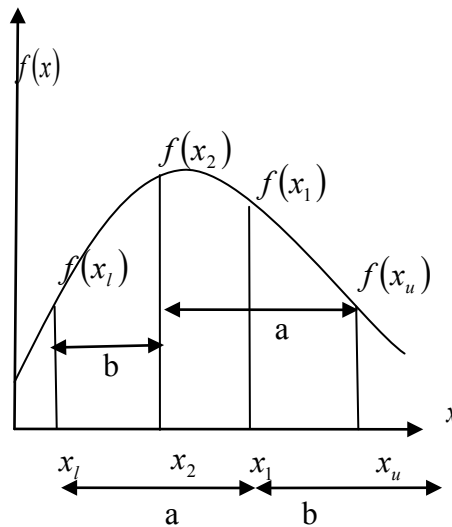


Figure 6 Intermediate points and their relation to boundary points

What happens after choosing the first two intermediate points?

Next we determine a new and smaller interval where the maximum value of the function lies in. We know that the new interval is either $[x_l, x_2, x_1]$ or $[x_2, x_1, x_u]$. To determine which of these intervals will be considered in the next iteration, the function is evaluated at the intermediate points x_2 and x_1 . If $f(x_2) > f(x_1)$, then the new region of interest will be $[x_l, x_2, x_1]$; else if $f(x_2) < f(x_1)$, then the new region of interest will be $[x_2, x_1, x_u]$. In Figure 6, we see that $f(x_2) > f(x_1)$, therefore our new region of interest is $[x_l, x_2, x_1]$. We should point out that the boundaries of the new smaller region are now determined by x_l and x_1 , and we already have one of the intermediate points, namely x_2 , conveniently located at a point where the ratio of the distance to the boundaries is the Golden Ratio. All that is left to do is to determine the location of the second intermediate point. Can you determine if the second point will be closer to x_l or x_1 ? This process of determining a new smaller region of interest and a new intermediate point will continue until the distance between the boundary points are sufficiently small.

The Golden Section Search Algorithm

The following algorithm can be used to determine the maximum of a function $f(x)$.

Initialization:

Determine x_l and x_u which is known to contain the maximum of the function $f(x)$.

Step 1

Determine two intermediate points x_1 and x_2 such that

$$x_1 = x_l + d$$

$$x_2 = x_u - d$$

where

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

Step 2

Evaluate $f(x_1)$ and $f(x_2)$.

If $f(x_1) > f(x_2)$, then determine new x_l, x_1, x_2 and x_u as shown in Equation set (5). Note that the only new calculation is done to determine the new x_1 .

$$x_l = x_2$$

$$x_2 = x_1$$

$$x_u = x_u$$

(5)

$$x_1 = x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

If $f(x_1) < f(x_2)$, then determine new x_l, x_1, x_2 and x_u as shown in Equation set (6). Note that the only new calculation is done to determine the new x_2 .

$$x_l = x_l$$

$$x_u = x_1$$

$$x_1 = x_2$$

(6)

$$x_2 = x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

Step 3

If $x_u - x_l < \varepsilon$ (a sufficiently small number), then the maximum occurs at $\frac{x_u + x_l}{2}$ and stop iterating, else go to Step 2.

Further Remarks and Explanation About The Golden Section Search Algorithm

The above discussion has assumed that the user can determine x_l and x_u which is known to contain the maximum of the function $f(x)$. In this section, the Golden Section algorithm is re-examined from a more rigorous viewpoint, and with the following 2 primary objectives:

- (a) Developing an automated procedure to determine the appropriated initial guesses for the lower and upper bounds, respectively.

- (b) Proving (in a more rigorous way) that we only need to find/compute only 1 (not 2) intermediate point, based on the current bracket.

To start the Golden Section search process, a small (positive) parameter “ δ ” is defined by the user, say “ δ ” = 0.05. The function value $g(\alpha = \delta) = g_1$ is initially computed. The second interval will be 1.618 times the previous (or first) interval (or $1.618 * \delta$), therefore, the next computed function value $g(\alpha = 2.618 * \delta) = g_2$ is computed.

Since g_2 is smaller than the previous value g_1 [see Figure 6A], one continues to consider the third interval which will be 1.618 times the second interval (or $1.618 * 1.618 \delta = 1.618^2 \delta$), and the next computed function value $g(\alpha = 5.232 * \delta) = g_3$ is computed. As indicated in Figure 6A, $\alpha = (5.232 * \delta)$ is also labeled as the (j-2)-th point on the curve!

Since g_3 is still smaller than the previous value g_2 [see Figure 6A], one continues to consider the fourth interval which will be 1.618 times the third interval (or $1.618 * 1.618^2 \delta = 1.618^3 \delta$) and the next computed function value $g(\alpha = 9.468 * \delta) = g_4$ is computed. As indicated in Figure 6A, $\alpha = (9.468 * \delta)$ is also labeled as the (j-1)-th point on the curve!

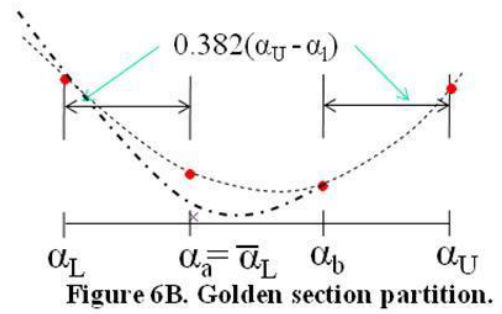
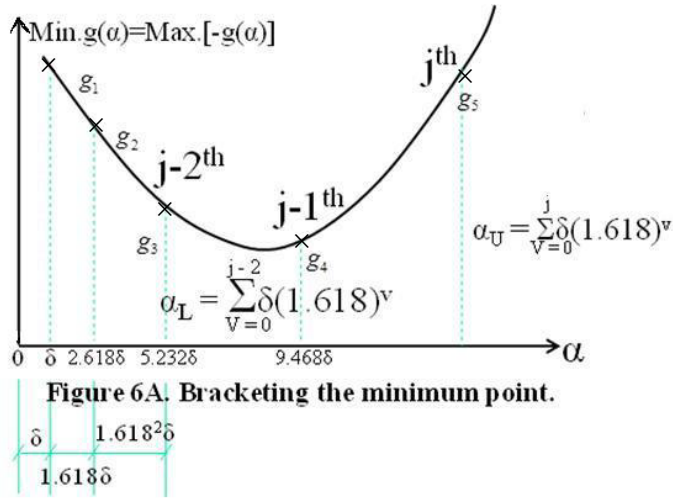
Since g_4 is still smaller than the previous value g_3 [see Figure 6A], one continues to consider the fifth interval which will be 1.618 times the fourth interval (or $1.618 * 1.618^3 \delta = 1.618^4 \delta$), and the next computed function value $g(\alpha = 16.3215 * \delta) = g_5$ is computed. As indicated in Figure 6A, $\alpha = (16.3215 * \delta)$ is also labeled as the j-th point on the curve!

At this moment, since $g_5 = g(\text{at the j-th point})$ is larger than $g_4 = g(\text{at the j-1-th point})$, the “decreasing pattern” is no longer true, therefore, we can establish the initial lower bound and the initial upper bound to be equal to the values of α at the (j-2)-th location and at the j-th location, respectively !

Based on the above observation and analysis, one can easily figure out the general formulas to compute and identify the initial lower and upper bound values for α as indicated in Figure 6A.

Having found the initial lower and upper bounds for α , the 2 intermediate points α_a and α_b need be inserted (with the same distance measured from the lower bound and upper bound, respectively) as shown in Figure 6B. Using Figure 6B, α_a can be computed and displayed as shown in equation (7).

Finally, with trivial algebraic manipulations, the value for α_a can be shown to be the same as the value for α (at the j-1_th point), as indicated in equation (8).



$$\alpha_a = \alpha_L + 0.382(\alpha_U - \alpha_L) = \sum_{v=0}^{j-2} \delta(1.618)^v + 0.382\delta(1.618)^{j-1}(1 + 1.618) \quad (7)$$

$$\alpha_a = \sum_{v=0}^{j-2} \delta(1.618)^v + 1\delta(1.618)^{j-1} = \sum_{v=0}^{j-1} \delta(1.618)^v = \text{already known!} \quad (8)$$

Based on Figures 6A and 6B, one observes that

- If $g(\alpha_a) = g(\alpha_b)$ then the minimum will be between α_b and α_b .
- If $g(\alpha_a) < g(\alpha_b)$ as shown in Figure 6B, then minimum will be between α_a and α_u .
Hence, $\bar{\alpha}_L = \text{new lower bound} = \alpha_a$
- Notice that: $\bar{\alpha}_u - \bar{\alpha}_L = \alpha_u - \alpha_a = \delta(1.618)^j$

and

$$\begin{aligned} \alpha_b - \bar{\alpha}_L &= \alpha_b - \alpha_a = (1 - 2 \times 0.382)(\alpha_U - \alpha_L) = (0.236)(\delta[1.618]^{j-1} + \delta[1.618]^j) \\ &= (0.236)(\delta[1.618]^{j-1} \times [1 + 1.618]) = 0.618(\delta[1.618]^{j-1}) \times \frac{1.618}{1.618} \\ \alpha_b - \bar{\alpha}_L &= (0.382) \times (\delta[1.618]^j) = 0.382(\bar{\alpha}_U - \bar{\alpha}_L) \end{aligned}$$

Thus α_b (with respect to α_u and α_L) plays the same role as α_a (with respect to α_u and α_L)!!

The step-by-step Golden Section procedure can be summarized as:

Step 1:

For a chosen small step size δ in α say, $\delta = 0.05$, let j be the smallest integer such that

$$g\left(\sum_{V=0}^j \delta(1.618)^V\right)g\left(\sum_{V=0}^{j-1} \delta(1.618)^V\right)$$

The upper and lower bound on α^i are $\alpha_U = \sum_{V=0}^j \delta(1.618)^V$ and $\alpha_L = \sum_{V=0}^{j-2} \delta(1.618)^V$.

Step 2:

Compute $g(\alpha_b)$, where $\alpha_a = \alpha_L + 0.382(\alpha_U - \alpha_L)$, and $\alpha_b = \alpha_L + 0.618(\alpha_U - \alpha_L)$

Note that $\alpha_a = \sum_{V=0}^{j-1} \delta(1.618)^V$, so $g(\alpha_a)$ is already known.

Step 3:

Compare $g(\alpha_a)$ and $g(\alpha_b)$ and go to Step 4, 5, or 6.

Step 4:

If $g(\alpha_a) < g(\alpha_b)$, then $\alpha_L \leq \alpha^i \leq \alpha_b$. By choice of α_a and α_b , the new points $\overline{\alpha}_L = \alpha_L$ and $\overline{\alpha}_u = \alpha_b$ have $\overline{\alpha}_b = \alpha_a$.

Compute $g(\overline{\alpha}_a)$, where $\overline{\alpha}_a = \overline{\alpha}_L + 0.382(\overline{\alpha}_u - \overline{\alpha}_L)$ and go to Step 7.

Step 5:

If $g(\alpha_a) > g(\alpha_b)$, then $\alpha_a \leq \alpha^i \leq \alpha_u$. Similar to the procedure in Step 4, put $\overline{\alpha}_L = \alpha_a$ and $\overline{\alpha}_u = \alpha_b$.

Compute $g(\overline{\alpha}_b)$, where $\overline{\alpha}_b = \overline{\alpha}_L + 0.618(\overline{\alpha}_u - \overline{\alpha}_L)$ and go to Step 7.

Step 6:

If $g(\alpha_a) = g(\alpha_b)$ put $\alpha_L = \alpha_a$ and $\alpha_u = \alpha_b$ and return to Step 2.

Step 7:

If $\overline{\alpha}_u - \overline{\alpha}_L$ is suitably small, put $\alpha^i = \frac{1}{2}(\overline{\alpha}_u + \overline{\alpha}_L)$ and stop.

Otherwise, delete the bar symbols on $\overline{\alpha}_L, \overline{\alpha}_a, \overline{\alpha}_b$ and $\overline{\alpha}_u$ and return to Step 3.

Example 2

Consider Figure 7 below. The cross-sectional area A of a gutter with equal base and edge length of 2 is given by

$$A = 4 \sin \theta (1 + \cos \theta)$$

Find the angle θ which maximizes the cross-sectional area of the gutter. Using an initial interval of $[0, \pi/2]$, find the solution after 2 iterations. Use an initial $\varepsilon = 0.05$.

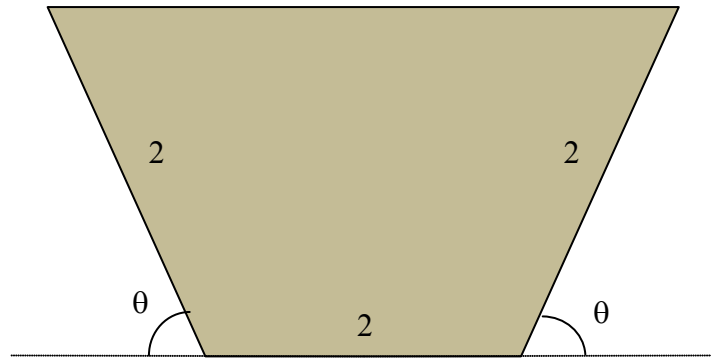


Figure 7 Cross section of the gutter

Solution

The function to be maximized is

$$f(\theta) = 4 \sin \theta (1 + \cos \theta)$$

Iteration 1:

Given the values for the boundaries of $x_l = 0$ and $x_u = \pi/2$, we can calculate the initial intermediate points as follows:

$$\begin{aligned} x_1 &= x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l) \\ &= 0 + \frac{\sqrt{5}-1}{2}(1.5708) \\ &= 0.97080 \\ x_2 &= x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l) \\ &= 1.5708 - \frac{\sqrt{5}-1}{2}(1.5708) \\ &= 0.60000 \end{aligned}$$

The function is evaluated at the intermediate points as $f(0.9708) = 5.1654$ and $f(0.60000) = 4.1227$. Since $f(x_1) > f(x_2)$, we eliminate the region to the left of x_2 and update the lower boundary point as $x_l = x_2$. The upper boundary point x_u remains unchanged. The second intermediate point x_2 is updated to assume the value of x_1 and finally the first intermediate point x_1 is re-calculated as follows:

$$\begin{aligned}
 x_1 &= x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l) \\
 &= 0.60000 + \frac{\sqrt{5}-1}{2}(1.5708 - 0.60000) \\
 &= 1.2000
 \end{aligned}$$

To check the stopping criteria the difference between x_u and x_l is calculated to be

$$x_u - x_l = 1.5708 - 0.60000 = 0.97080$$

which is greater than $\varepsilon = 0.05$. The process is repeated in the second iteration.

Iteration 2:

The values for the boundary and intermediate points used in this iteration were calculated in the previous iteration as shown below.

$$x_l = 0.60000$$

$$x_u = 1.5708$$

$$x_1 = 1.2000$$

$$x_2 = 0.97080$$

Again the function is evaluated at the intermediate points as $f(1.20000) = 5.0791$ and $f(0.97080) = 5.1654$. Since $f(x_1) < f(x_2)$, the opposite of the case seen in the first iteration, we eliminate the region to the right of x_1 and update the upper boundary point as $x_u = x_1$. The lower boundary point x_l remains unchanged. The first intermediate point x_1 is updated to assume the value of x_2 and finally the second intermediate point x_2 is recalculated as follows:

$$\begin{aligned}
 x_2 &= x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l) \\
 &= 1.2000 - \frac{\sqrt{5}-1}{2}(1.2000 - 0.60000) \\
 &= 0.82918
 \end{aligned}$$

To check the stopping criteria the difference between x_u and x_l is calculated to be

$$\begin{aligned}
 x_u - x_l &= 1.2000 - 0.60000 \\
 &= 0.60000
 \end{aligned}$$

which is greater than $\varepsilon = 0.05$. At the end of the second iteration the solution is

$$\frac{x_u + x_l}{2} = \frac{1.2000 + 0.60000}{2} \\ = 0.90000$$

Therefore, the maximum area occurs when $\theta = 0.9$ radians or 51.6° .

The iterations will continue until the stopping criterion is met. Summary results of all the iterations are shown in Table 1. Note that at the end of the 9th iteration, $\varepsilon < 0.05$ which causes the search to stop. The optimal value is calculated as the average of the upper and lower boundary points.

$$\frac{x_u + x_l}{2} = \frac{1.0249 + 1.0583}{2} \\ = 1.0416$$

which is about 59.68° . The area of the gutter at this angle is $f(1.0416) = 5.1960$. The theoretical optimal solution to the problem happens at exactly 60° which is 1.0472 radians and an area of 5.1962.

Table 1 Summary of iterations for Example 1

Iteration	x_l	x_u	x_1	x_2	$f(x_1)$	$f(x_2)$	ε
1	0.00000	1.5708	0.97081	0.59999	5.1654	4.1226	1.5708
2	0.59999	1.5708	1.2000	0.97081	5.0791	5.1654	0.97081
3	0.59999	1.2000	0.97081	0.82917	5.1654	4.9418	0.59999
4	0.82917	1.2000	1.0583	0.97081	5.1955	5.1654	0.37081
5	0.97081	1.2000	1.1124	1.0583	5.1743	5.1955	0.22918
6	0.97081	1.1124	1.0583	1.0249	5.1955	5.1936	0.14164
7	1.0249	1.1124	1.0790	1.0583	5.1909	5.1955	0.08754
8	1.0249	1.0790	1.0583	1.0456	5.1955	5.1961	0.05410
9	1.0249	1.0583	1.0456	1.0377	5.1961	5.1957	0.03344

OPTIMIZATION

Topic	Golden Search Method
Summary	Textbook notes for the golden search method
Major	All engineering majors
Authors	Ali Yalcin, Autar Kaw
Date	December 19, 2012
Web Site	http://numericalmethods.eng.usf.edu
