Chapter 07.04
Romberg Rule of Integration

After reading this chapter, you should be able to:

1. *derive the Romberg rule of integration, and*
2. *use the Romberg rule of integration to solve problems.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Romberg rule of approximating integrals of the form

$$I = \int_a^b f(x)dx \tag{1}$$

where

$f(x)$ is called the integrand

$a =$ lower limit of integration

$b =$ upper limit of integration

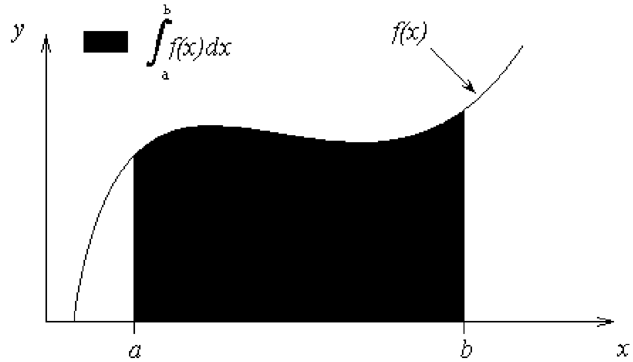

Figure 1 Integration of a function.

Error in Multiple-Segment Trapezoidal Rule

The true error obtained when using the multiple segment trapezoidal rule with n segments to approximate an integral

$$\int_a^b f(x)dx$$

is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \qquad (2)$$

where for each i, ξ_i is a point somewhere in the domain $[a+(i-1)h, a+ih]$, and

the term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$. This leads us to say that the true error E_t in Equation (2) is approximately proportional to

$$E_t \approx \alpha \frac{1}{n^2} \qquad (3)$$

for the estimate of $\int_a^b f(x)dx$ using the n-segment trapezoidal rule.

Table 1 shows the results obtained for

$$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

using the multiple-segment trapezoidal rule.

Table 1 Values obtained using multiple segment trapezoidal rule for
$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$.

n	Approximate Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
1	11868	-807	7.296	---
2	11266	-205	1.854	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

The true error for the 1-segment trapezoidal rule is -807, while for the 2-segment rule, the true error is -205. The true error of -205 is approximately a quarter of -807. The true error gets approximately quartered as the number of segments is doubled from 1 to 2. The same trend is observed when the number of segments is doubled from 2 to 4 (the true error for 2-segments is -205 and for four segments is -51.5). This follows Equation (3).
This information, although interesting, can also be used to get a better approximation of the integral. That is the basis of Richardson's extrapolation formula for integration by the trapezoidal rule.

Richardson's Extrapolation Formula for Trapezoidal Rule

The true error, E_t, in the n-segment trapezoidal rule is estimated as

$$E_t \approx \alpha \frac{1}{n^2}$$

$$E_t \approx \frac{C}{n^2} \tag{4}$$

where C is an approximate constant of proportionality.
Since

$$E_t = TV - I_n \tag{5}$$

where

TV = true value

I_n = approximate value using n-segments

Then from Equations (4) and (5),

$$\frac{C}{n^2} \approx TV - I_n \tag{6}$$

If the number of segments is doubled from n to $2n$ in the trapezoidal rule,

$$\frac{C}{(2n)^2} \approx TV - I_{2n} \tag{7}$$

Equations (6) and (7) can be solved simultaneously to get

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \tag{8}$$

Example 1

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Romberg's rule to find the distance covered. Use the 2-segment and 4-segment trapezoidal rule results given in Table 1.
b) Find the true error for part (a).
c) Find the absolute relative true error for part (a).

Solution

a)
$$I_2 = 11266 \text{ m}$$
$$I_4 = 11113 \text{ m}$$

Using Richardson's extrapolation formula for the trapezoidal rule, the true value is given by

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$\begin{aligned} TV &\approx I_4 + \frac{I_4 - I_2}{3} \\ &= 11113 + \frac{11113 - 11266}{3} \\ &= 11062 \text{ m} \end{aligned}$$

b) The exact value of the above integral is

$$\begin{aligned} x &= \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \\ &= 11061 \text{ m} \end{aligned}$$

so the true error

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 11061 - 11062 \\ &= -1 \text{ m} \end{aligned}$$

c) The absolute relative true error, $|\in_t|$, would then be

$$\begin{aligned} |\in_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{11061 - 11062}{11061} \right| \times 100 \\ &= 0.00904\% \end{aligned}$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, and 8 segments. Results are compared with those of the trapezoidal rule.

Table 2 Values obtained using Richardson's extrapolation formula for the trapezoidal rule for

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt\ .$$

n	Trapezoidal Rule	$\lvert\in_t\rvert\%$ for Trapezoidal Rule	Richardson's Extrapolation	$\lvert\in_t\rvert\%$ for Richardson's Extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Romberg Integration

Romberg integration is the same as Richardson's extrapolation formula as given by Equation (8). However, Romberg used a recursive algorithm for the extrapolation as follows.

The estimate of the true error in the trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

Since the segment width, h, is given by

$$h = \frac{b-a}{n}$$

Equation (2) can be written as

$$E_t = -\frac{h^2(b-a)}{12}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \qquad (9)$$

The estimate of true error is given by

$$E_t \approx Ch^2 \qquad (10)$$

It can be shown that the exact true error could be written as

$$E_t = A_1h^2 + A_2h^4 + A_3h^6 + \ldots \qquad (11)$$

and for small h,

$$E_t = A_1h^2 + O(h^4) \qquad (12)$$

Since we used $E_t \approx Ch^2$ in the formula (Equation (12)), the result obtained from Equation (10) has an error of $O(h^4)$ and can be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n}-I_n}{3}$$

$$= I_{2n} + \frac{I_{2n}-I_n}{4^{2-1}-1} \qquad (13)$$

where the variable TV is replaced by $\left(I_{2n}\right)_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by the sign =.
Hence the estimate of the true value now is

$$TV \approx \left(I_{2n}\right)_R + Ch^4$$

Determine another integral value with further halving the step size (doubling the number of segments),

$$\left(I_{4n}\right)_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3} \tag{14}$$

then

$$TV \approx \left(I_{4n}\right)_R + C\left(\frac{h}{2}\right)^4$$

From Equation (13) and (14),

$$\begin{aligned} TV &\approx \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{15} \\ &= \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{4^{3-1} - 1} \end{aligned} \tag{15}$$

The above equation now has the error of $O\left(h^6\right)$. The above procedure can be further improved by using the new values of the estimate of the true value that has the error of $O\left(h^6\right)$ to give an estimate of $O\left(h^8\right)$.

Based on this procedure, a general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1},\ k \geq 2 \tag{16}$$

The index k represents the order of extrapolation. For example, $k=1$ represents the values obtained from the regular trapezoidal rule, $k=2$ represents the values obtained using the true error estimate as $O\left(h^2\right)$, etc. The index j represents the more and less accurate estimate of the integral. The value of an integral with a $j+1$ index is more accurate than the value of the integral with a j index.

For $k=2$, $j=1$,

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} \\ &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \end{aligned}$$

For $k=3$, $j=1$,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1}$$

$$= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \tag{17}$$

Example 2

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment trapezoidal rule results as given in Table 1.

Solution

From Table 1, the needed values from the original the trapezoidal rule are

$$I_{1,1} = 11868$$
$$I_{1,2} = 11266$$
$$I_{1,3} = 11113$$
$$I_{1,4} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment trapezoidal rule, respectively. To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$
$$= 11266 + \frac{11266 - 11868}{3}$$
$$= 11065$$

Similarly

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$
$$= 11113 + \frac{11113 - 11266}{3}$$
$$= 11062$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$
$$= 11074 + \frac{11074 - 11113}{3}$$
$$= 11061$$

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$
$$= 11062 + \frac{11062 - 11065}{15}$$
$$= 11062$$

Similarly

$$
\begin{aligned}
I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\
&= 11061 + \frac{11061 - 11062}{15} \\
&= 11061
\end{aligned}
$$

For the third order extrapolation values,

$$
\begin{aligned}
I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\
&= 11061 + \frac{11061 - 11062}{63} \\
&= 11061\,\mathrm{m}
\end{aligned}
$$

Table 3 shows these increasingly correct values in a tree graph.

Table 3 Improved estimates of the value of an integral using Romberg integration.

		First Order	Second Order	Third Order
1-segment	11868			
		11065		
2-segment	11266		11062	
		11062		11061
4-segment	11113		11061	
		11061		
8-segment	11074			

INTEGRATION	
Topic	Romberg Rule
Summary	Textbook notes of Romberg Rule of integration.
Major	General Engineering
Authors	Autar Kaw
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Web Site	http://numericalmethods.eng.usf.edu