

Newton's Divided Difference Polynomial Method of Interpolation

Major: All Engineering Majors

Authors: Autar Kaw, Jai Paul

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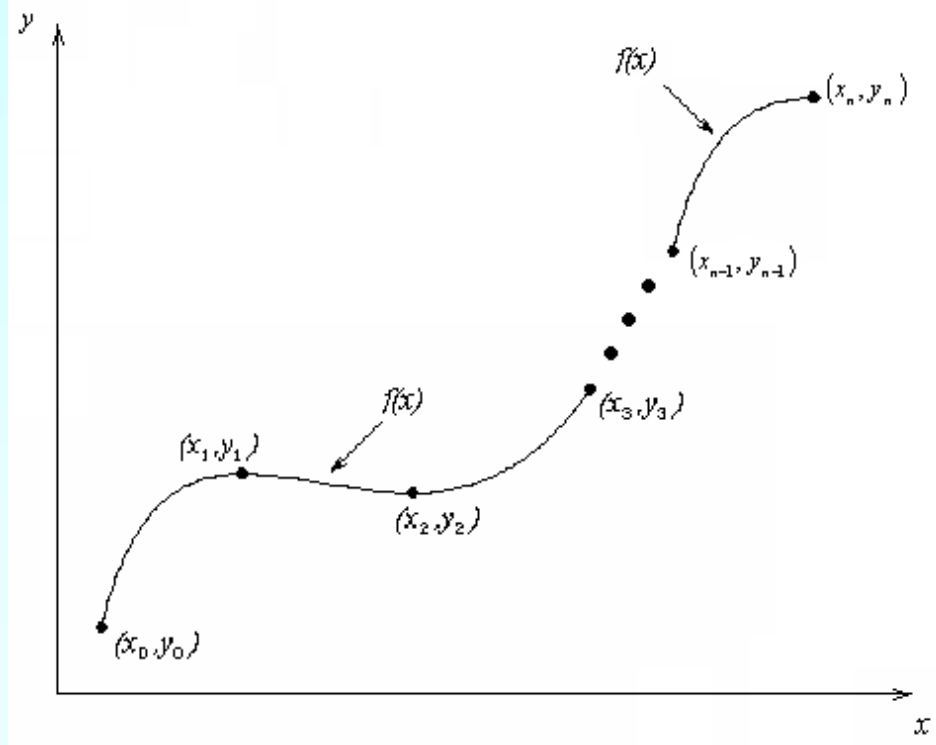
Transforming Numerical Methods Education for STEM
Undergraduates

Newton's Divided Difference Method of Interpolation

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What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

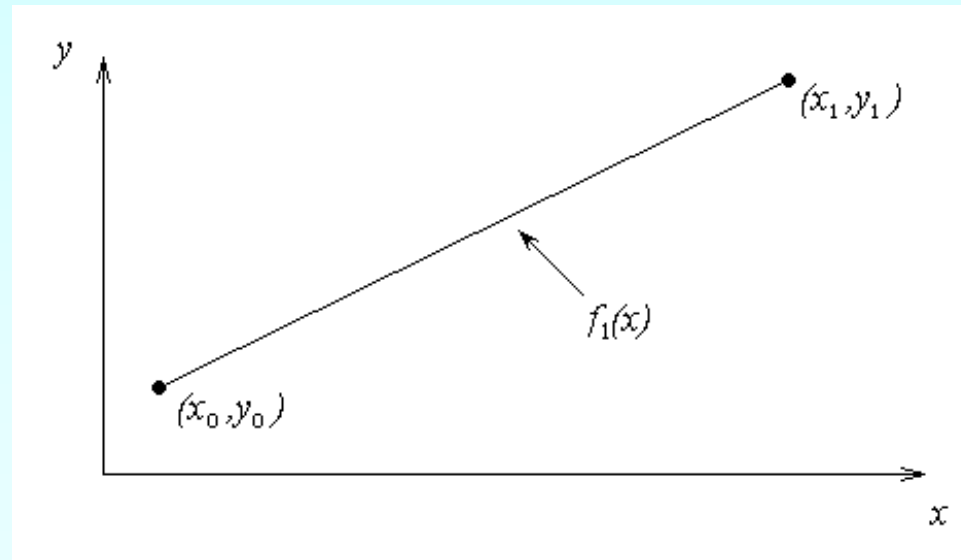
Linear interpolation: Given (x_0, y_0) , (x_1, y_1) , pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

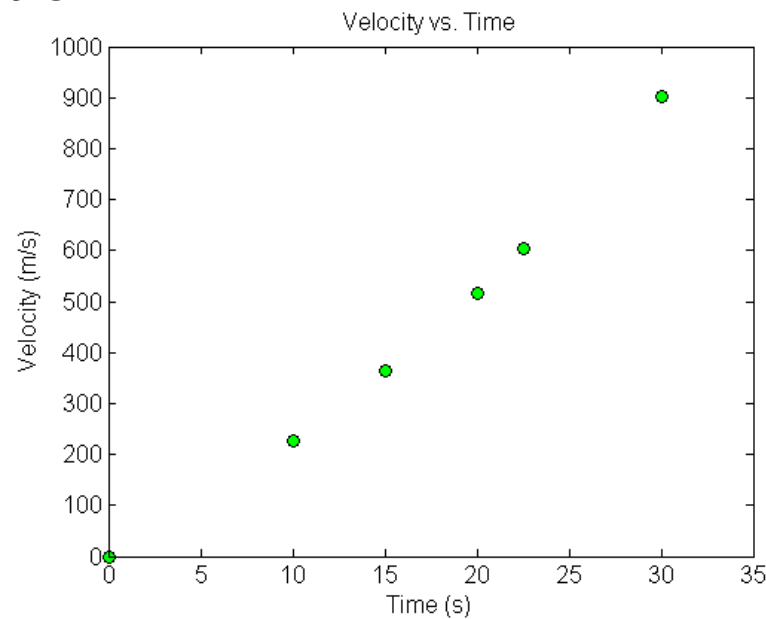


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

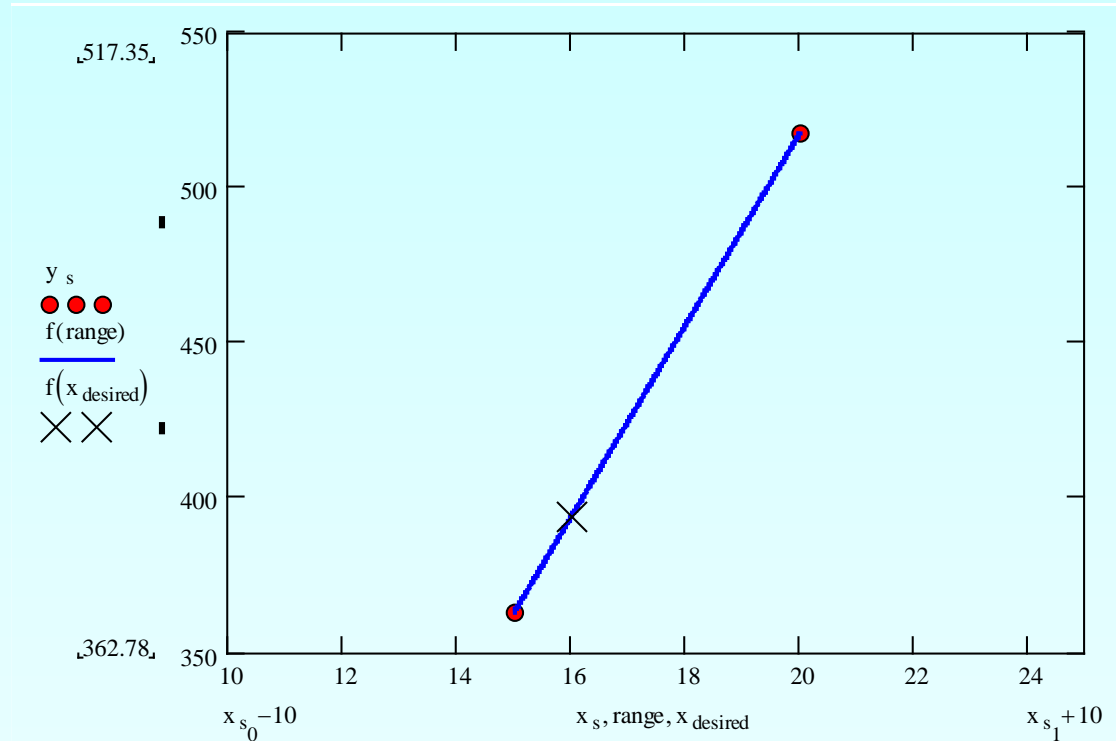
$$v(t) = b_0 + b_1(t - t_0)$$

$$t_0 = 15, v(t_0) = 362.78$$

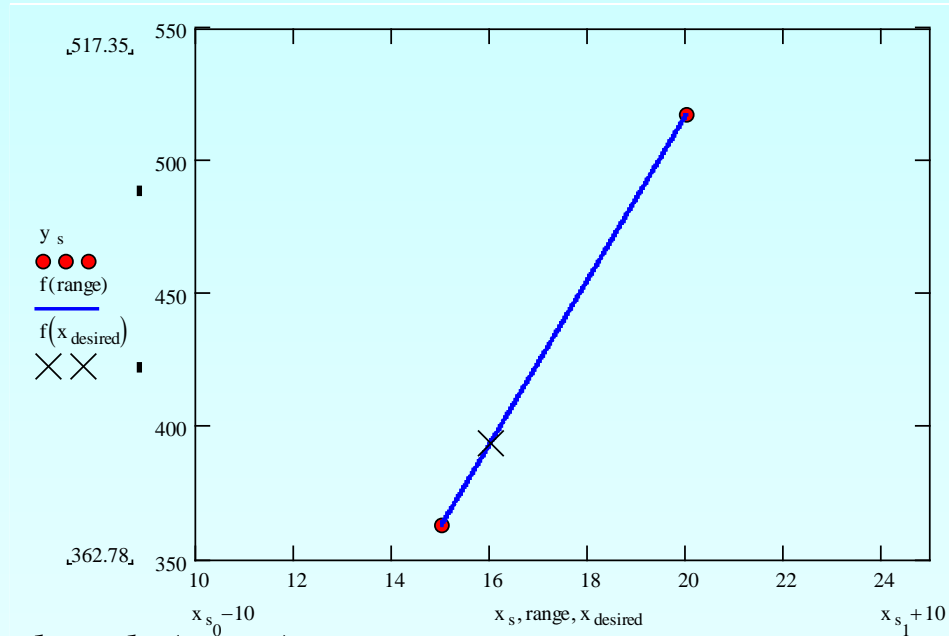
$$t_1 = 20, v(t_1) = 517.35$$

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$



Linear Interpolation (contd)



$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15), 15 \leq t \leq 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Quadratic Interpolation

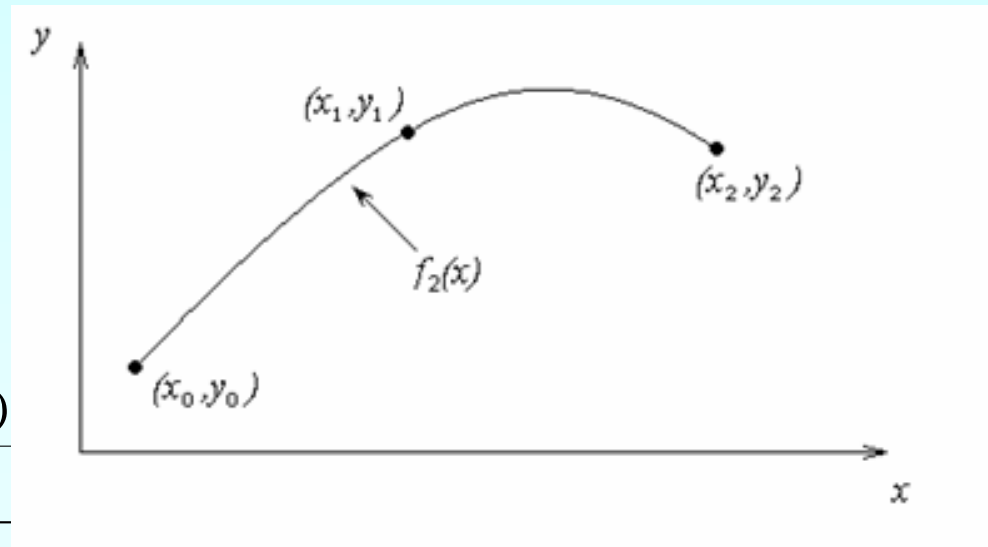
Given (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) , fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

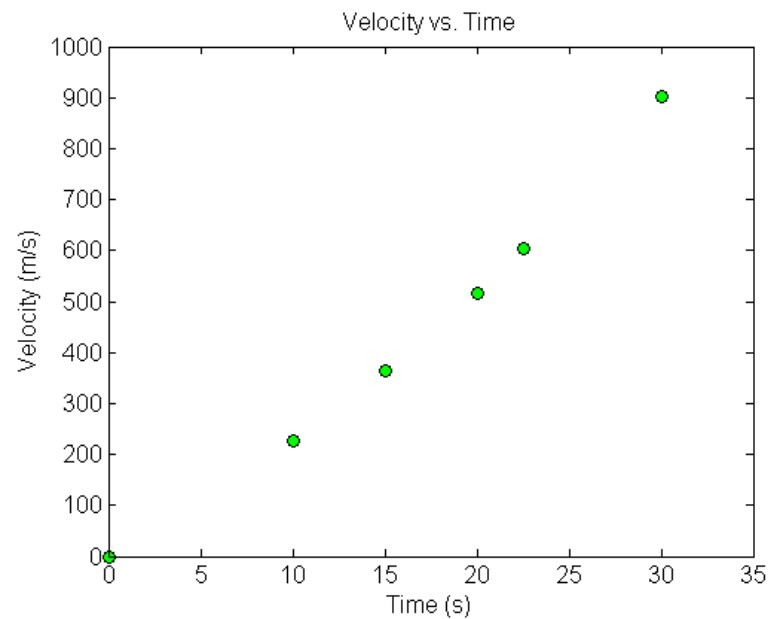
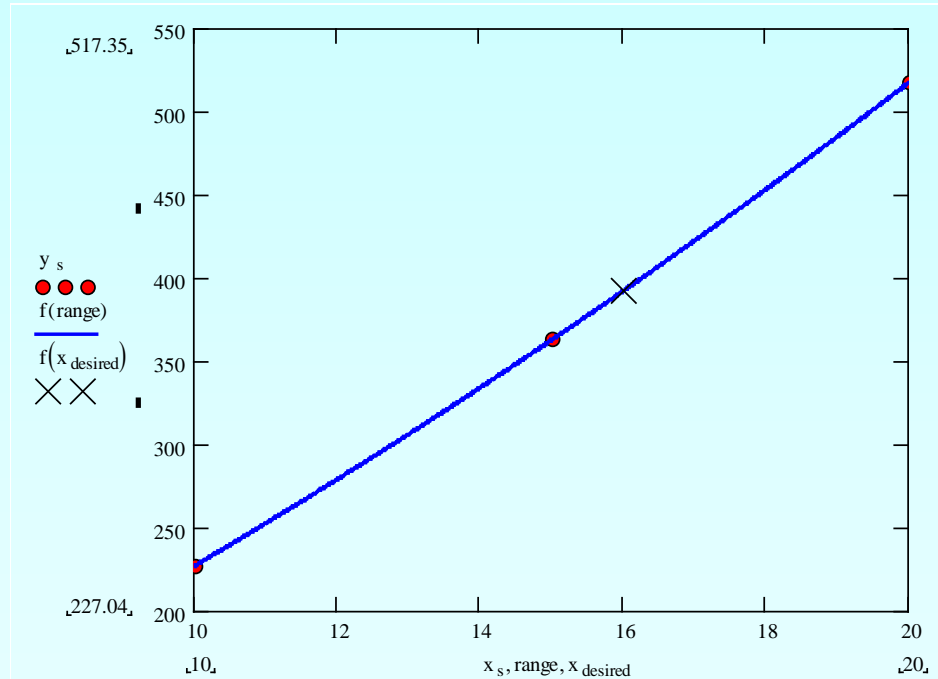


Figure. Velocity vs. time data for the rocket example

Quadratic Interpolation (contd)



$$t_0 = 10, v(t_0) = 227.04$$

$$t_1 = 15, v(t_1) = 362.78$$

$$t_2 = 20, v(t_2) = 517.35$$

Quadratic Interpolation (contd)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\frac{v(t_2) - v(t_1)}{t_2 - t_1} - \frac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\frac{517.35 - 362.78}{20 - 15} - \frac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Quadratic Interpolation (contd)

$$\begin{aligned}v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) \\ &= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \quad 10 \leq t \leq 20\end{aligned}$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first order and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.19 - 393.69}{392.19} \right| \times 100 \\ &= 0.38502 \%\end{aligned}$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n + 1)$ data points, $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

\vdots

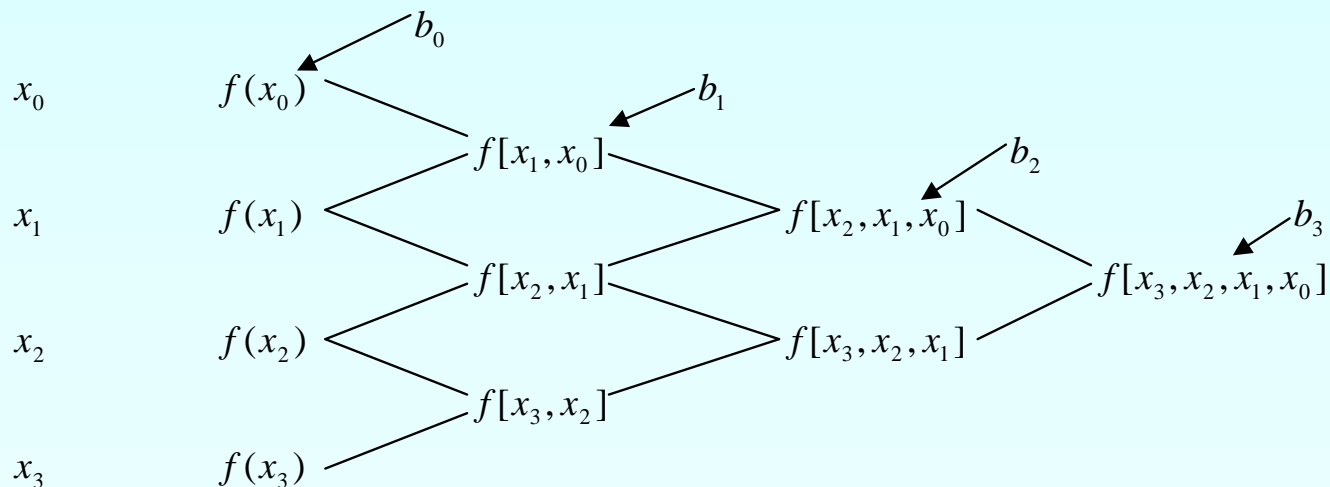
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$

General form

The third order polynomial, given (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using the Newton Divided Difference method for cubic interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

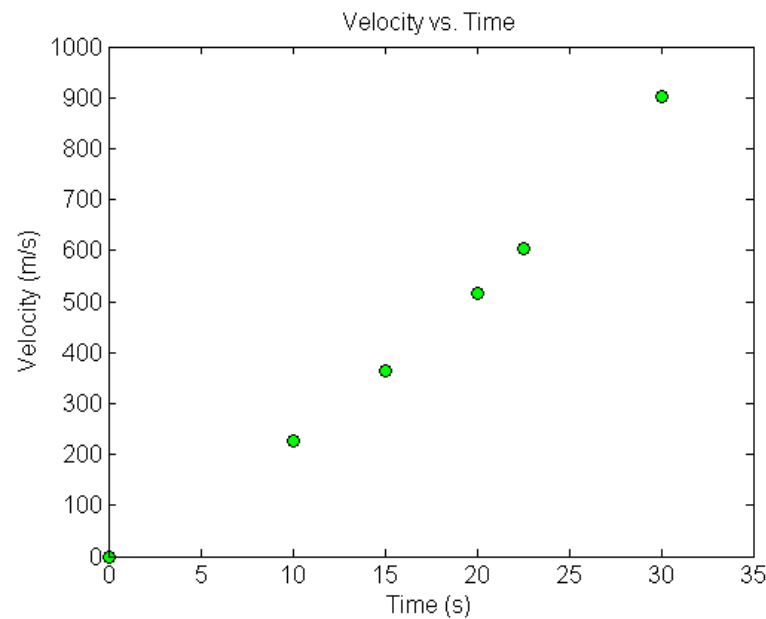


Figure. Velocity vs. time data for the rocket example

Example

The velocity profile is chosen as

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

we need to choose four data points that are closest to $t = 16$

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

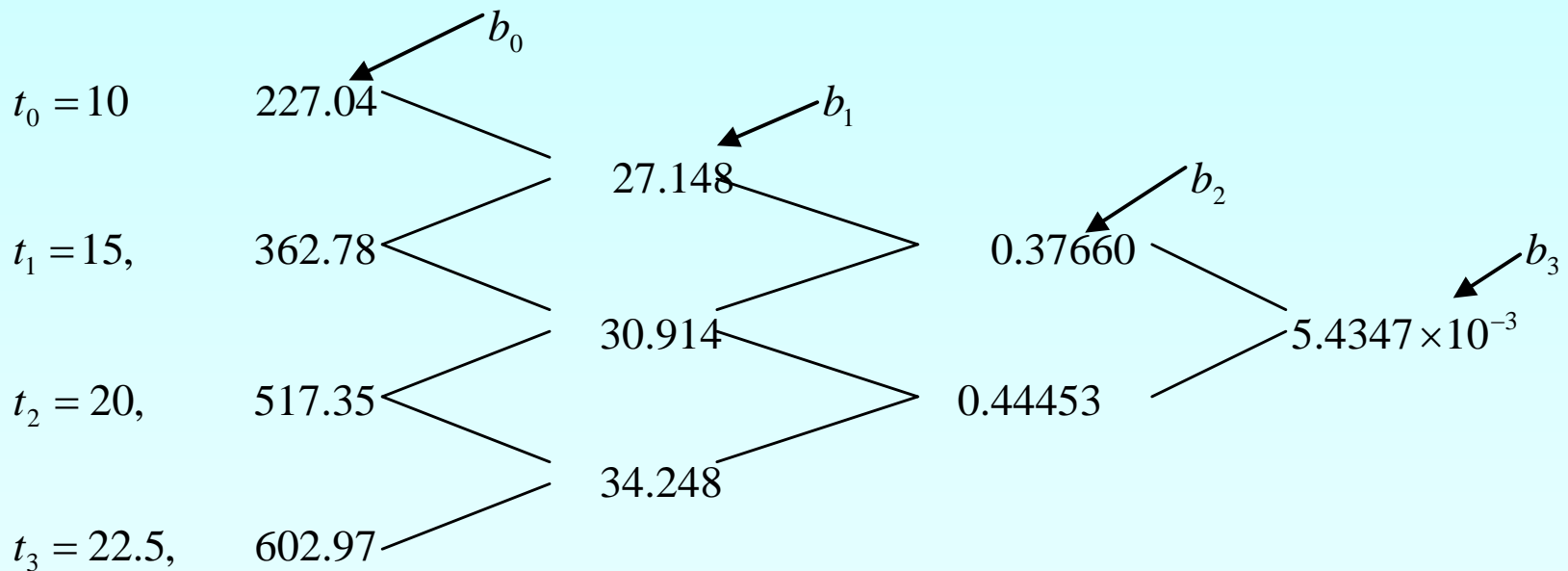
$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

The values of the constants are found as:

$$b_0 = 227.04; \quad b_1 = 27.148; \quad b_2 = 0.37660; \quad b_3 = 5.4347 \times 10^{-3}$$

Example



$$b_0 = 227.04; \quad b_1 = 27.148; \quad b_2 = 0.37660; \quad b_3 = 5.4347 \times 10^{-3}$$

Example

Hence

$$\begin{aligned}v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2) \\&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\&\quad + 5.4347 * 10^{-3}(t - 10)(t - 15)(t - 20)\end{aligned}$$

At $t = 16$,

$$\begin{aligned}v(16) &= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\&\quad + 5.4347 * 10^{-3}(16 - 10)(16 - 15)(16 - 20) \\&= 392.06 \text{ m/s}\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\&= 0.033427 \%\end{aligned}$$

Comparison Table

Order of Polynomial	1	2	3
$v(t=16)$ m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	-----	0.38502 %	0.033427 %

Distance from Velocity Profile

Find the distance covered by the rocket from $t=11s$ to $t=16s$?

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) + 5.4347 * 10^{-3} (t - 10)(t - 15)(t - 20) \quad 10 \leq t \leq 22.5$$

$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3 \quad 10 \leq t \leq 22.5$$

So

$$\begin{aligned} s(16) - s(11) &= \int_{11}^{16} v(t) dt \\ &= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3) dt \\ &= \left[-4.2541t + 21.265 \frac{t^2}{2} + 0.13204 \frac{t^3}{3} + 0.0054347 \frac{t^4}{4} \right]_{11}^{16} \end{aligned}$$

$$= 1605 \text{ m}$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at $t=16s$ given that

$$v(t) = -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)$$

$$= 21.265 + 0.26408t + 0.016304t^2$$

$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.664 \text{ m/s}^2$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

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