Chapter 02.03
Differentiation of Discrete Functions

After reading this chapter, you should be able to:

1. *find approximate values of the first derivative of functions that are given at discrete data points, and*
2. *use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

Forward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

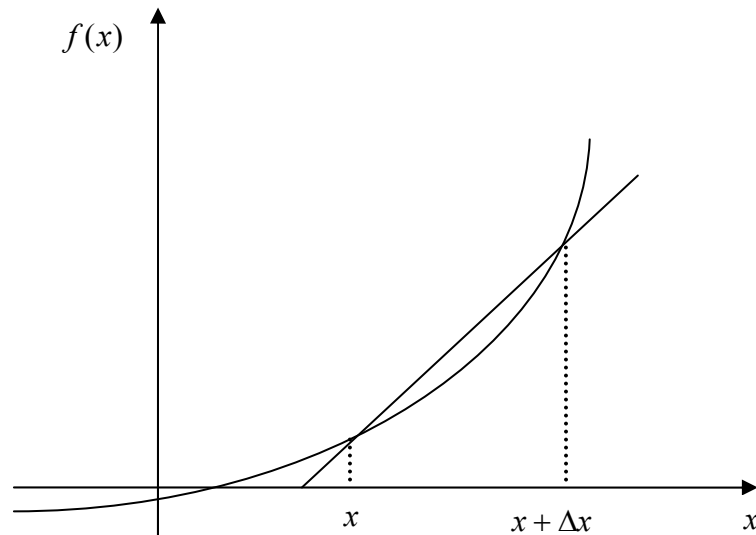$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Figure 1 Graphical representation of forward difference approximation of first derivative.

So given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, the value of $f'(x)$ for $x_i \leq x \leq x_{i+1}$, $i = 0, ..., n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Using forward divided difference, find the acceleration of the rocket at $t = 16$ s.

Solution

To find the acceleration at $t = 16$ s, we need to choose the two values of velocity closest to $t = 16$ s, that also bracket $t = 16$ s to evaluate it. The two points are $t = 15$ s and $t = 20$ s

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 15$$

$$t_{i+1} = 20$$

$$\Delta t = t_{i+1} - t_i$$

$$= 20 - 15$$

$$= 5$$

$$a(16) \approx \frac{v(20) - v(15)}{5}$$

$$= \frac{517.35 - 362.78}{5}$$

$$= 30.914 \text{ m/s}^2$$

Direct Fit Polynomials

In this method, given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a nth order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots + (n-1)a_{n-1} x^{n-2} + na_n x^{n-1}$$

Similarly, other derivatives can also be found.

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Using a third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16 \text{ s}$.

Solution

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the velocity at $t = 16\,\text{s}$, and we are using a third order polynomial, we need to choose the four points closest to $t = 16$ and that also bracket $t = 16$ to evaluate it. The four points are $t_0 = 10, t_1 = 15, t_2 = 20$ and $t_3 = 22.5$.

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

such that

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$
$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$
$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$
$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 10 & 100 & 1000 \\ 1 & 15 & 225 & 3375 \\ 1 & 20 & 400 & 8000 \\ 1 & 22.5 & 506.25 & 11391 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$
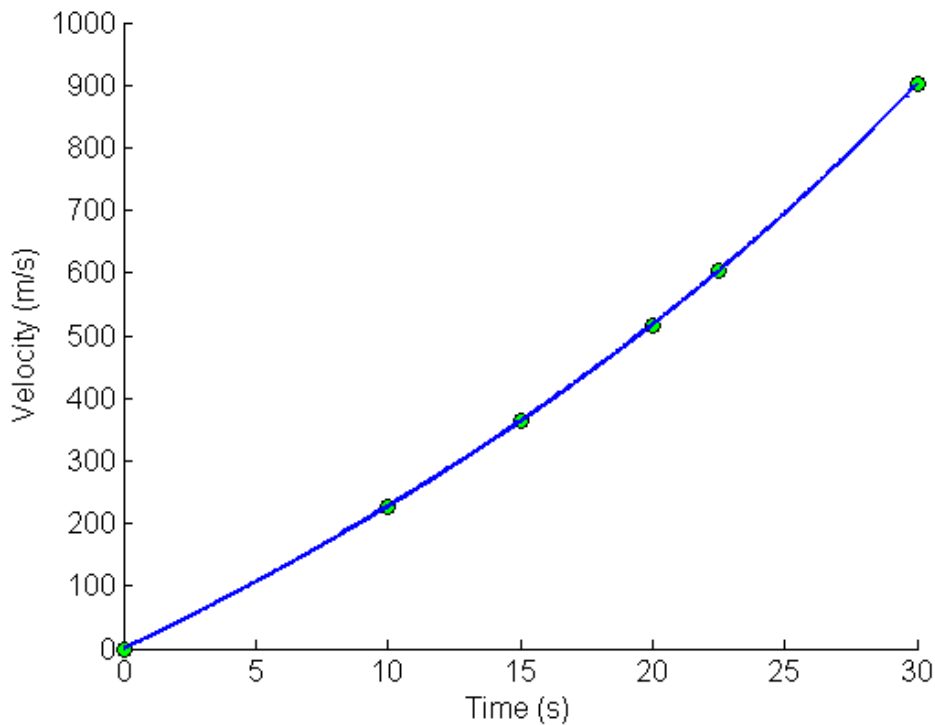

Figure 2 Graph of upward velocity of the rocket vs. time.

Solving the above four equations gives

$$a_0 = -4.3810$$
$$a_1 = 21.289$$
$$a_2 = 0.13065$$
$$a_3 = 0.0054606$$

Hence

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt} v(t) \Big|_{t=16}$$

Given that $v(t) = -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$,

$$a(t) = \frac{d}{dt} v(t)$$
$$= \frac{d}{dt} \left(-4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3 \right)$$
$$= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \le t \le 22.5$$
$$a(16) = 21.289 + 0.26130(16) + 0.016382(16)^2$$
$$= 29.664 \text{ m/s}^2$$

Lagrange Polynomial

In this method, given $(x_0, y_0), \ldots, (x_n, y_n)$, one can fit a nth order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the nth order polynomial that approximates the function $y = f(x)$ and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1),$ and (x_2, y_2) is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the acceleration at $t = 16\,\text{s}$ using second order Lagrangian polynomial interpolation for velocity.

Solution

$$v(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right) v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right) v(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right) v(t_2)$$

$$a(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)} v(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)} v(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)} v(t_2)$$

$$a(16) = \frac{2(16)-(15+20)}{(10-15)(10-20)}(227.04) + \frac{2(16)-(10+20)}{(15-10)(15-20)}(362.78)$$

$$+ \frac{2(16)-(10+15)}{(20-10)(20-15)}(517.35)$$

$$= -0.06(227.04) - 0.08(362.78) + 0.14(517.35)$$

$$= 29.784\,\text{m/s}^2$$

DIFFERENTIATION	
Topic	Differentiation of Discrete Functions
Summary	These are textbook notes differentiation of discrete functions
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