# Chapter 08.03
# Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
# Mechanical Engineering

**Example 1**

A solid steel shaft at room temperature of $27\ ^{\circ}\text{C}$ is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at $-33\ ^{\circ}\text{C}$. The rate of change of temperature of the solid shaft $\theta$ is given by

$$\frac{d\theta}{dt} = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 + \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$\theta(0) = 27\ ^{\circ}\text{C}$$

Using the Runge-Kutta 2nd order method, find the temperature of the steel shaft after 86400 seconds. Take a step size of $h = 43200$ seconds.

**Solution**

$$\frac{d\theta}{dt} = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$f(t,\theta) = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

Per Heun's method

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(t_i, \theta_i)$$

$$k_2 = f(t_i + h, \theta_i + k_1 h)$$

For $i = 0,\ t_0 = 0,\ \theta_0 = 27$

$$k_1 = f(t_0, \theta_0)$$

$$k_1 = f(0, 27)$$

$$= \left(-5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}(27)^4 + 2.33\times 10^{-5}(27)^3 \\ +1.35\times 10^{-3}(27)^2 + 5.42\times 10^{-2}(27) + 5.588 \end{pmatrix}(27+33)\right)$$

$$= -0.0020893$$

$$k_2 = f(t_0 + h, \theta_0 + k_1 h)$$

$$= f(0+43200, 27+(-0.0020893)43200)$$
$$= f(43200, -63.278)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-63.278)^4+2.33\times10^{-5}(-63.278)^3\\+1.35\times10^{-3}(-63.278)^2+5.42\times10^{-2}(-63.278)+5.588\end{pmatrix}(-63.278+33)\right)$$
$$= -0.0092607$$

$$\theta_1 = \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 27 + \left(\frac{1}{2}(-0.0020893) + \frac{1}{2}(-0.0092607)\right)43200$$
$$= 27 + (-0.0056750)43200$$
$$= -218.16\ ^\circ\text{C}$$

$\theta_1$ is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\ \text{s}$$
$$\theta(43200) \approx \theta_1 = -218.16\ ^\circ\text{C}$$

For $i = 1$, $t_1 = 43200$, $\theta_1 = -218.16$

$$k_1 = f(t_1, \theta_1)$$
$$= f(43200, -218.16)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-218.16)^4+2.33\times10^{-5}(-218.16)^3\\+1.35\times10^{-3}(-218.16)^2+5.42\times10^{-2}(-218.16)+5.588\end{pmatrix}(-218.16+33)\right)$$
$$= -8.4304$$

$$k_2 = f(t_1 + h, \theta_1 + k_1 h)$$
$$= f(43200+43200, -218.16+(-8.4304)43200)$$
$$= f(86400, -364410)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-364410)^4+2.33\times10^{-5}(-364410)^3\\+1.35\times10^{-3}(-364410)^2+5.42\times10^{-2}(-364410)+5.588\end{pmatrix}(-364410+33)\right)$$
$$= -1.2638\times10^{17}$$

$$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= -218.16 + \left(\frac{1}{2}(-8.4304) + \frac{1}{2}(-1.2638\times10^{17})\right)43200$$
$$= -218.16 + (-6.3190\times10^{16})43200$$
$$= -2.7298\times10^{21}\ ^\circ\text{C}$$

$\theta_2$ is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$
$$\theta(86400) \approx \theta_2 = -2.7298 \times 10^{21} \text{ °C}$$

The solution to this nonlinear equation at $t = 86400\text{s}$ is

$$\theta(86400) = -26.099 \text{ °C}$$

The results from Heun's method are compared with exact results in Figure 1.

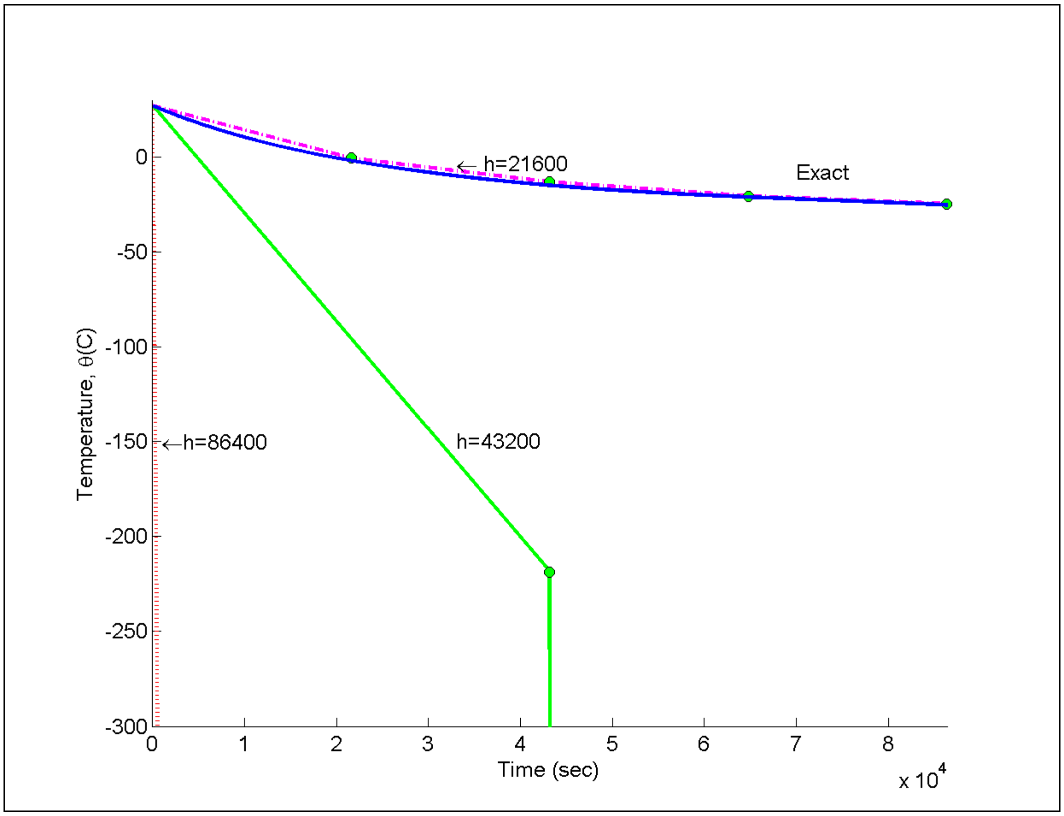


**Figure 1** Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2 below.

**Table 1** Effect of step size for Heun's method.

| Step size, $h$ | $\theta(86400)$ | $E_t$ | $\lvert\in_t\rvert\%$ |
|---|---|---|---|
| 86400 | $-58466$ | 58440 | 223920 |
| 43200 | $-2.7298\times10^{21}$ | $2.7298\times10^{21}$ | $1.0460\times10^{22}$ |
| 21600 | $-24.537$ | $-1.5619$ | 5.9845 |
| 10800 | $-25.785$ | $-0.31368$ | 1.2019 |
| 5400 | $-26.027$ | $-0.072214$ | 0.27670 |

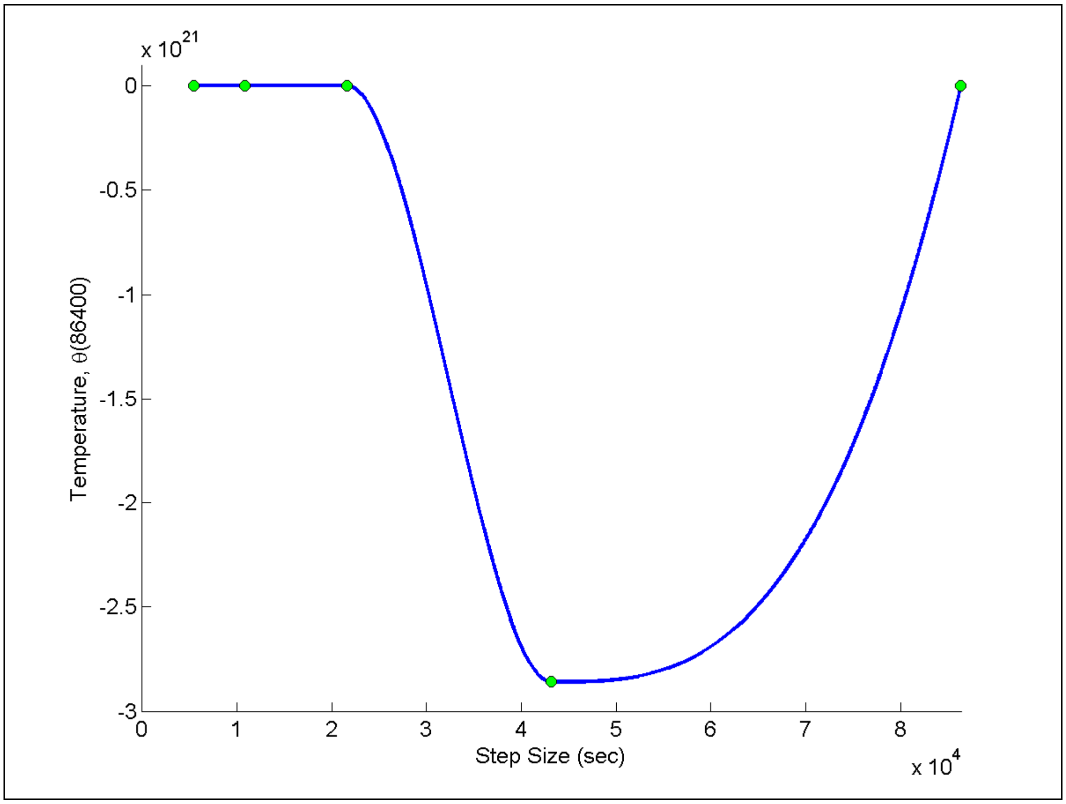


**Figure 2** Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2$^{nd}$ order method results are shown as a function of step size.

**Table 2** Comparison of Euler and the Runge-Kutta methods.

| Step size, $h$ | $\theta(86400)$ | | | |
|---|---|---|---|---|
| | Euler | Heun | Midpoint | Ralston |
| 86400 | −153.52 | −58466 | −774.64 | −12163 |
| 43200 | −464.32 | $-2.7298\times10^{21}$ | 0.33691 | −19.776 |
| 21600 | −29.541 | −24.537 | −24.069 | −24.268 |
| 10800 | −27.795 | −25.785 | −25.808 | −25.777 |
| 5400 | −26.958 | −26.027 | −26.039 | −26.032 |

While in Figure 3, the comparison is shown over the range of time.

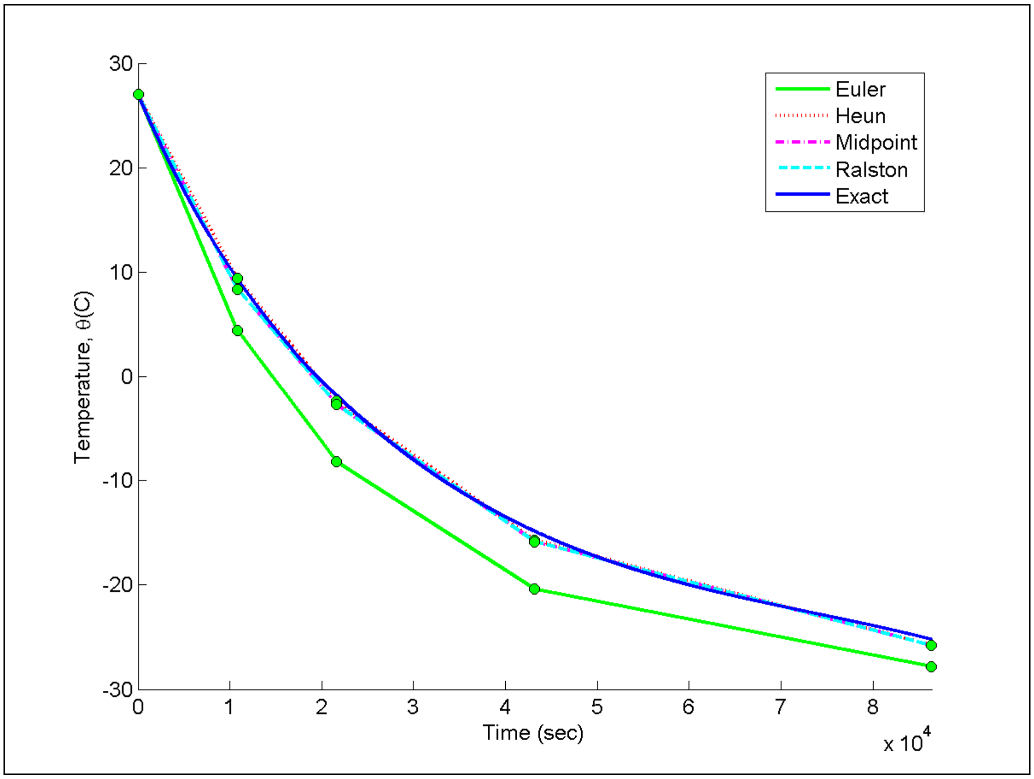


**Figure 3** Comparison of Euler and Runge Kutta methods with exact results over time.