Chapter 08.02
Euler's Method for Ordinary Differential Equations-More Examples
Mechanical Engineering

Example 1

A solid steel shaft at room temperature of $27\ ^{o}\text{C}$ is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at $-33\ ^{o}\text{C}$. The rate of change of temperature of the solid shaft θ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ + 5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$\theta(0) = 27\ ^{\circ}\text{C}$$

Using Euler's method, find the temperature of the steel shaft after 86400 seconds. Take a step size of $h = 43200$ seconds.

Solution

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ + 5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$f(t,\theta) = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ + 5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

The Euler's method reduces to

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 27$

$$\begin{aligned}
\theta_1 &= \theta_0 + f(t_0,\theta_0)h \\
&= 27 + f(0,27)43200 \\
&= 27 + \left(-5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}(27)^4 + 2.33\times10^{-5}(27)^3 \\ +1.35\times10^{-3}(27)^2 + 5.42\times10^{-2}(27) + 5.588 \end{pmatrix}(27+33)\right)43200 \\
&= 27 + (-0.0020893)43200 \\
&= -63.258\ ^{\circ}\text{C}
\end{aligned}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\ \text{s}$$

$$\theta(43200) \approx \theta_1 = -63.258\ ^{\circ}\text{C}$$

For $i = 1$, $t_1 = 43200$, $\theta_1 = -63.258$

$$\theta_2 = \theta_1 + f(t_1, \theta_1)h$$
$$= -63.258 + f(43200, -63.258)43200$$
$$= -63.258 + \left(-5.33 \times 10^{-6} \begin{pmatrix} -3.69 \times 10^{-6}(-63.258)^4 + 2.33 \times 10^{-5}(-63.258)^3 \\ +1.35 \times 10^{-3}(-63.258)^2 \\ +5.42 \times 10^{-2}(-63.258) + 5.588 \end{pmatrix} (-63.258 + 33) \right) 43200$$
$$= -63.258 + (-0.0092607)43200$$
$$= -463.32\ ^\circ\text{C}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$
$$\theta(86400) \approx \theta_2 = -463.32\ ^\circ\text{C}$$

Figure 1 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 43200$.

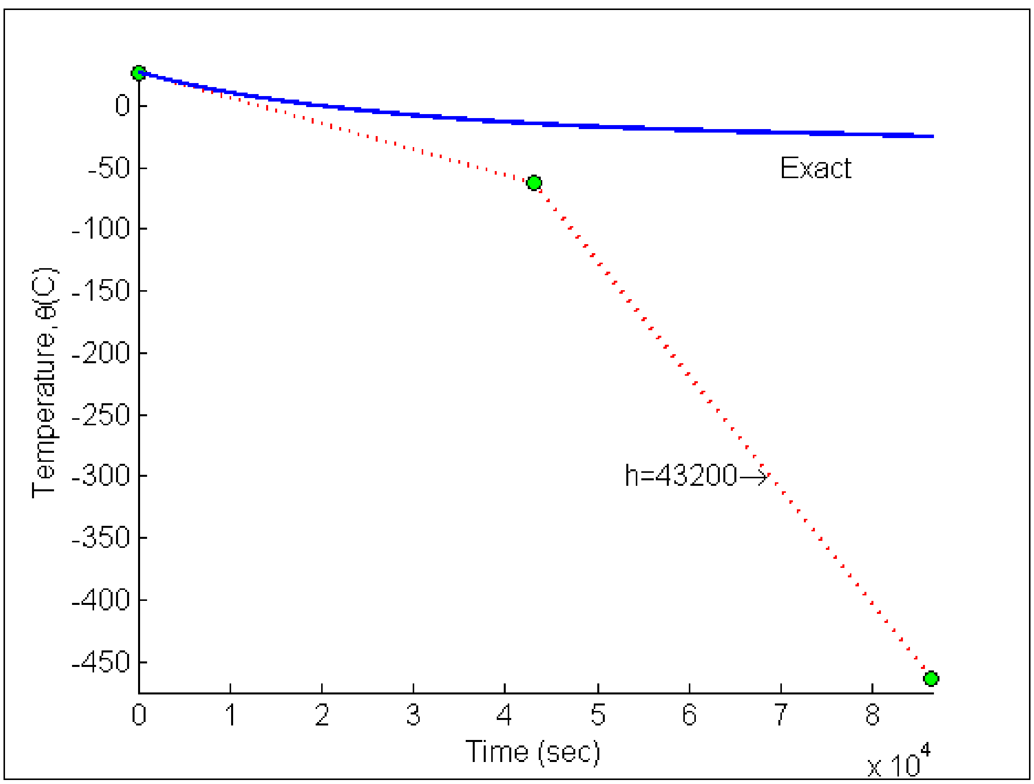

Figure 1 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Temperature at 86400 seconds as a function of step size, h.

Step size, h	$\theta(86400)$	E_t	$\lvert\epsilon_t\rvert\%$
86400	-153.52	127.42	488.21
43200	-463.32	437.22	1675.2
21600	-29.542	3.4421	13.189
10800	-27.795	1.6962	6.4988
5400	-26.958	0.85870	3.2902

Figure 2 shows how the temperature varies as a function of time for different step sizes.

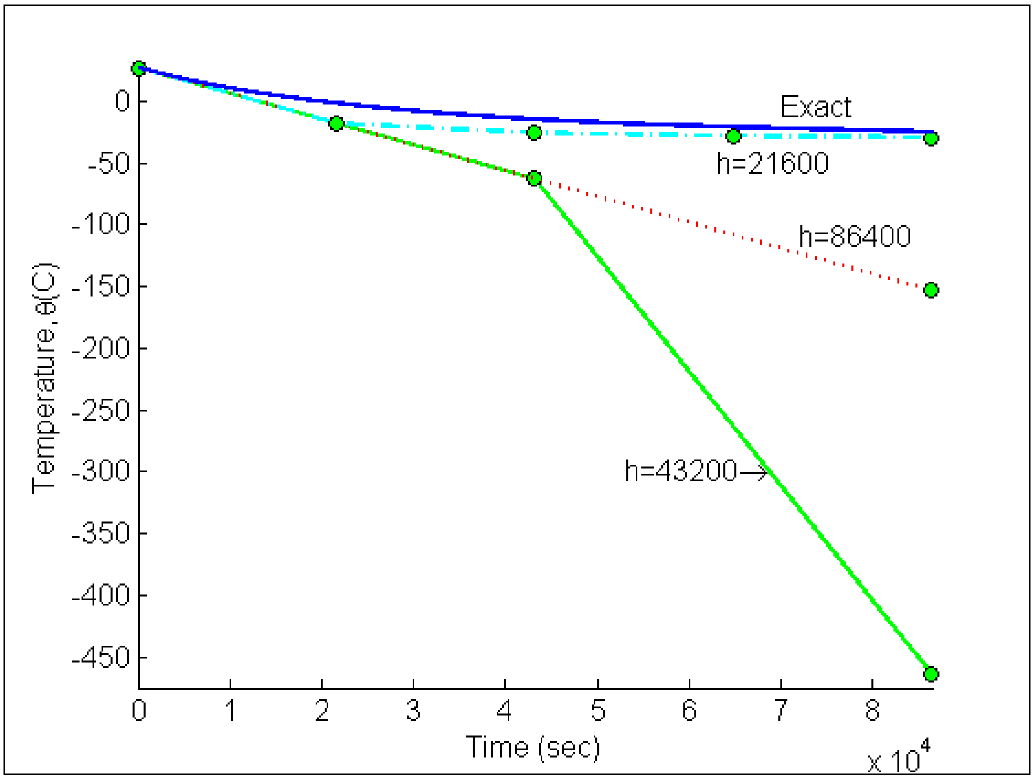

Figure 2 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated temperature at $t = 86400\,\text{s}$ as a function of step size are plotted in Figure 3.

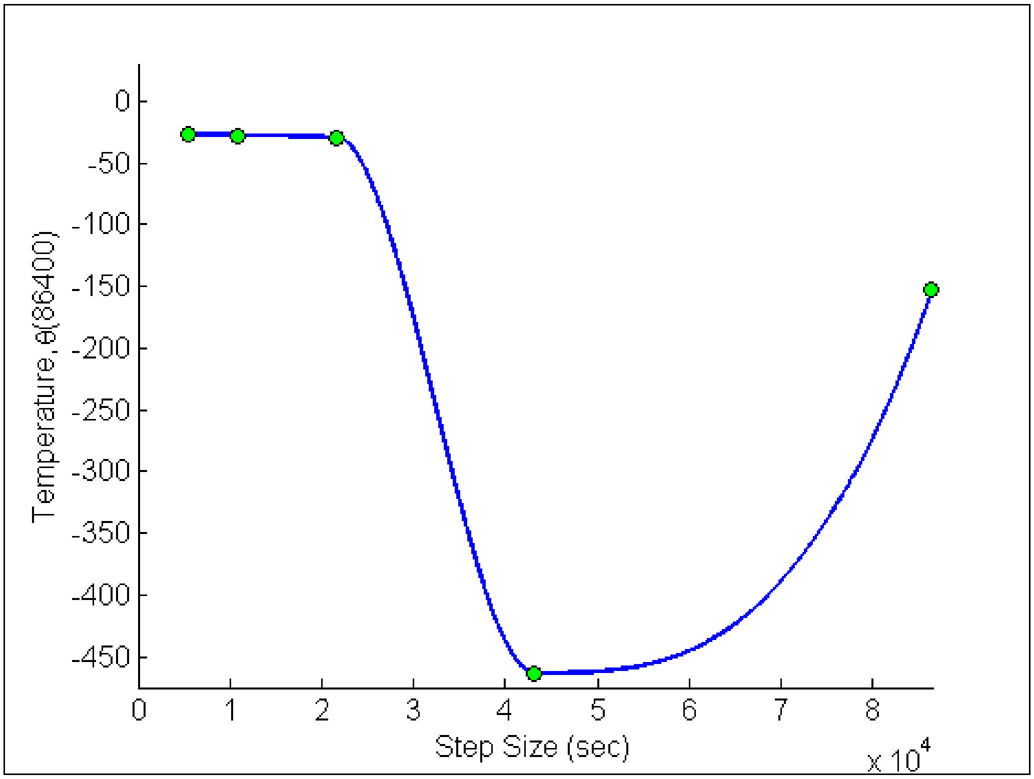

Figure 3 Effect of step size in Euler's method.

The solution to this nonlinear equation at $t = 86400\text{s}$ is

$$\theta(86400) = -26.099^\circ\text{C}$$