Chapter 05.05
Spline Method of Interpolation – More Examples
Mechanical Engineering

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T (°F)	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

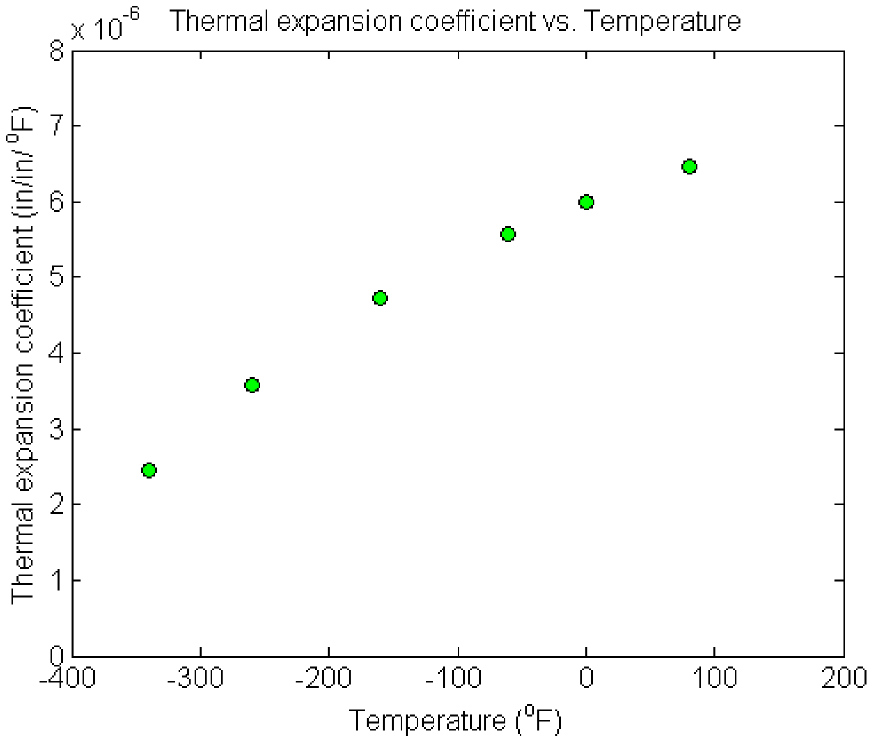

Figure 1 Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14\,°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14\,°\text{F}$ using linear splines.

Solution

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$ and we are using linear splines, we need to choose the two data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. The two points are $T_0 = 0$ and $T_1 = -60\,°\text{F}$.

Then

$$T_0 = 0,\ \alpha(T_0) = 6.00 \times 10^{-6}$$
$$T_1 = -60,\ \alpha(T_1) = 5.58 \times 10^{-6}$$

gives

$$\alpha(T) = \alpha(T_0) + \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}(T - T_0)$$
$$= 6.00 \times 10^{-6} + \frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0}(T - 0)$$

Hence

$$\alpha(T) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(T - 0),\ -60 \le T \le 0$$

At $T = -14$,

$$\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14 - 0)$$

$= 5.902 \times 10^{-6}$ in/in/°F

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, $\text{T}(°\text{F})$	Thermal Expansion Coefficient, $\alpha(\text{in/in/°F})$
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

a) Determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using quadratic splines. Find the absolute relative approximate error for the second order approximation.

b) The reduction of the diameter is given more accurately by

$$\Delta D = D\int_{T_r}^{T_f} \alpha dT$$

where $T_r =$ room temperature $(°\text{F})$

$T_f =$ temperature of cooling medium $(°\text{F})$

Given

$T_r = 80°\text{F}$

$T_f = -108°\text{F}$

find a better estimate. What is the difference between the value found in part (a) and part (b)?

Solution

a) Since there are six data points, five quadratic splines pass through them.

$$\begin{aligned}\alpha(T) &= a_1T^2 + b_1T + c_1, && -340 \le T \le -260\\ &= a_2T^2 + b_2T + c_2, && -260 \le T \le -160\\ &= a_3T^2 + b_3T + c_3, && -160 \le T \le -60\\ &= a_4T^2 + b_4T + c_4, && -60 \le T \le 0\\ &= a_5T^2 + b_5T + c_5, && 0 \le T \le 80\end{aligned}$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1T^2 + b_1T + c_1$ passes through $T = -340$ and $T = -260$.

$$a_1(-340)^2 + b_1(-340) + c_1 = 2.45\times10^{-6} \quad (1)$$

$$a_1(-260)^2 + b_1(-260) + c_1 = 3.58\times10^{-6} \quad (2)$$

$a_2T^2 + b_2T + c_2$ passes through $T = -260$ and $T = -160$.

$$a_2(-260)^2 + b_2(-260) + c_2 = 3.58\times10^{-6} \quad (3)$$

$$a_2(-160)^2 + b_2(-160) + c_2 = 4.72\times10^{-6} \quad (4)$$

$a_3T^2 + b_3T + c_3$ passes through $T = -160$ and $T = -60$.

$$a_3(-160)^2 + b_3(-160) + c_3 = 4.72\times10^{-6} \quad (5)$$

$$a_3(-60)^2 + b_3(-60) + c_3 = 5.58\times10^{-6}$$
(6)

$a_4T^2 + b_4T + c_4$ passes through $T = -60$ and $T = 0$.

$$a_4(-60)^2 + b_4(-60) + c_4 = 5.58\times10^{-6} \quad (7)$$

$$a_4(0)^2 + b_4(0) + c_4 = 6.00\times10^{-6} \quad (8)$$

$a_5T^2 + b_5T + c_5$ passes through $T = 0$ and $T = 80$.

$$a_5(0)^2 + b_5(0) + c_5 = 6.00\times10^{-6} \quad (9)$$

$$a_5(80)^2 + b_5(80) + c_5 = 6.47\times10^{-6} \quad (10)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $T = -260$

$$2a_1(-260) + b_1 - 2a_2(-260) - b_2 = 0 \quad (11)$$

At $T = -160$

$$2a_2(-160) + b_2 - 2a_3(-160) - b_3 = 0 \quad (12)$$

At $T = -60$

$$2a_3(-60) + b_3 - 2a_4(-60) - b_4 = 0 \quad (13)$$

At $T = 0$

$$2a_4(0) + b_4 - 2a_5(0) - b_5 = 0 \quad (14)$$

3. Assuming the first spline $a_1T^2 + b_1T + c_1$ is linear,

$$a_1 = 0 \qquad (15)$$

$$\begin{bmatrix}
1.156\times10^5 & -340 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6400 & 80 & 1\\
-520 & 1 & 0 & 520 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -320 & 1 & 0 & 320 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -120 & 1 & 0 & 120 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1\\ b_1\\ c_1\\ a_2\\ b_2\\ c_2\\ a_3\\ b_3\\ c_3\\ a_4\\ b_4\\ c_4\\ a_5\\ b_5\\ c_5 \end{bmatrix}
=
\begin{bmatrix} 2.45\times10^{-6}\\ 3.58\times10^{-6}\\ 3.58\times10^{-6}\\ 4.72\times10^{-6}\\ 4.72\times10^{-6}\\ 5.58\times10^{-6}\\ 5.58\times10^{-6}\\ 6.00\times10^{-6}\\ 6.00\times10^{-6}\\ 6.47\times10^{-6}\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

Solving the above 15 equations gives the 15 unknowns as

i	a_i	b_i	c_i
1	0	0.014125×10^{-6}	7.2525×10^{-6}
2	-2.725×10^{-11}	-4.5×10^{-11}	5.4104×10^{-6}
3	-7.5×10^{-13}	0.008435×10^{-6}	6.0888×10^{-6}
4	-2.5417×10^{-11}	0.005475×10^{-6}	6×10^{-6}
5	5×10^{-12}	0.005475×10^{-6}	6×10^{-6}

Therefore, the splines are given by

$$\begin{aligned}
\alpha(T) &= 0.014125\times10^{-6}T + 7.2525\times10^{-6}, && -340 \le T \le -260\\
&= -2.725\times10^{-11}T^2 - 4.5\times10^{-11}T + 5.4104\times10^{-6}, && -260 \le T \le -160\\
&= -7.5\times10^{-13}T^2 + 0.008435\times10^{-6}T + 6.0888\times10^{-6}, && -160 \le T \le -60\\
&= -2.5417\times10^{-11}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}, && -60 \le T \le 0\\
&= 5\times10^{-12}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}, && 0 \le T \le 80
\end{aligned}$$

At $T = -14$°F

$$\begin{aligned}
\alpha(-14) &= -2.5417\times10^{-11}(-14)^2 + 0.005475\times10^{-6}(-14) + 6\times10^{-6}\\
&= 5.9184\times10^{-6} \text{ in/in/°F}
\end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$\left|\in_a\right| = \left|\frac{5.9184\times10^{-6} - 5.902\times10^{-6}}{5.9184\times10^{-6}}\right|\times100$$
$$= 0.27657\%$$

b) The reduction of the diameter is given more accurately by

$$\Delta D = D\int_{T_r}^{T_f}\alpha dT$$

where

$T_r =$ room temperature $(°\text{F})$

$T_f =$ boiling temperature of liquid nitrogen $(°\text{F})$

Given

$T_r = 80\,°\text{F}$

$T_f = -108\,°\text{F}$

To find $\int_{T_r}^{T_f}\alpha dT$, we can integrate the quadratic splines with respect to temperature.

$$\int_{T_r}^{T_f}\alpha dT = \int_{80}^{-108}\alpha(T)dT$$
$$= \int_{-60}^{-108}\alpha(T)dT + \int_{0}^{-60}\alpha(T)dT + \int_{80}^{0}\alpha(T)dT$$
$$= \int_{-60}^{-108}\left(-7.5\times10^{-13}T^2 + 0.008435\times10^{-6}T + 6.0888\times10^{-6}\right)dT$$
$$+ \int_{0}^{-60}\left(-2.5417\times10^{-11}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}\right)dT$$
$$+ \int_{80}^{0}\left(5\times10^{-12}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}\right)dT$$
$$= \left[-7.5\times10^{-13}\frac{T^3}{3} + 0.008435\times10^{-6}\frac{T^2}{2} + 6.0888\times10^{-6}T\right]_{-60}^{-108}$$
$$+ \left[-2.5417\times10^{-11}\frac{T^3}{3} + 0.005475\times10^{-6}\frac{T^2}{2} + 6\times10^{-6}T\right]_{0}^{-60}$$
$$+ \left[5\times10^{-12}\frac{T^3}{3} + 0.005475\times10^{-6}\frac{T^2}{2} + 6\times10^{-6}T\right]_{80}^{0}$$
$$= [-257.99]\times10^{-6} + [-348.32]\times10^{-6} + [-498.373]\times10^{-6}$$
$$= -1104.7\times10^{-6}$$

$$\int_{T_r}^{T_f} \alpha dT = -1104.7 \times 10^{-6} \text{ in/in}$$

To compare this result with our results from part (a), we take the average coefficient of thermal expansion over this interval, given by:

$$\alpha_{avg} = \frac{\int_{T_r}^{T_f} \alpha dT}{T_f - T_r}$$

$$= \frac{-1104.7 \times 10^{-6}}{-108 - 80}$$

$$= 5.8760 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from part (a) and part (b) is

$$|\in_a| = \left| \frac{5.8760 \times 10^{-6} - 5.9184 \times 10^{-6}}{5.8760 \times 10^{-6}} \right| \times 100$$

$$= 0.72178\%$$