Chapter 02.03
Differentiation of Discrete Functions-More Examples Mechanical Engineering

Example 1

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 1.

Table 1 Coefficient of thermal expansion as a function of temperature.

Temperature, T ($°\text{F}$)	Coefficient of thermal expansion, α (in/in/$°\text{F}$)
80	6.47×10^{-6}
40	6.24×10^{-6}
-40	5.72×10^{-6}
-120	5.09×10^{-6}
-200	4.30×10^{-6}
-280	3.33×10^{-6}
-340	2.45×10^{-6}

(a) Is the rate of change of the coefficient of thermal expansion with respect to temperature more at $T = 80\ °\text{F}$ than at $T = -340\ °\text{F}$?

(b) The data given in Table 1 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80\ °\text{F}$ and at $T = -340\ °\text{F}$.

Solution

(a) Using the forward divided difference approximation method at $T = 80\ °\text{F}$,

$$\frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_{i+1}) - \alpha(T_i)}{\Delta T}$$

$$T_i = 80$$

$$\Delta T = -40$$

$$\begin{aligned} T_{i+1} &= T_i + \Delta T \\ &= 80 + (-40) \\ &= 40 \end{aligned}$$

$$\frac{d\alpha(80)}{dT} \approx \frac{\alpha(40)-\alpha(80)}{-40}$$
$$= \frac{6.24\times10^{-6} - 6.47\times10^{-6}}{-40}$$
$$= 5.75\times10^{-9} \text{ in/in/°F}^2$$

Using the backward divided difference approximation method at $T = -340\,°\text{F}$,

$$\frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_i)-\alpha(T_{i-1})}{\Delta T}$$
$$T_i = -340$$
$$\Delta T = -60$$
$$T_{i-1} = T_i - \Delta T$$
$$= -340-(-60)$$
$$= -280$$
$$\frac{d\alpha(-340)}{dT} \approx \frac{\alpha(-340)-\alpha(-280)}{-60}$$
$$= \frac{2.45\times10^{-6} - 3.33\times10^{-6}}{-60}$$
$$= 0.14667\times10^{-7} \text{ in/in/°F}^2$$

From the above two results it is clear that the rate of change of the coefficient of thermal expansion is more at $T = 80\,°\text{F}$ than $T = -340\,°\text{F}$.

b) Given:

$$\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2$$
$$\frac{d\alpha}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}T$$
$$\frac{d\alpha(80)}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}(80)$$
$$= 4.3246\times10^{-9} \text{ in/in/°F}^2$$

$$\frac{d\alpha(-340)}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}(-340)$$
$$= 0.14585\times10^{-7} \text{ in/in/°F}^2$$

Table 2 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	Divided Difference Approximation	2nd Order Polynomial Regression
80	5.75×10^{-9} in/in/°F^2	4.3246×10^{-9} in/in/°F^2
-340	0.14667×10^{-7} in/in/°F^2	0.14585×10^{-7} in/in/°F^2

Example 2

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 3.

Table 3 Coefficient of thermal expansion as a function of temperature.

Temperature, T ($^\circ$F)	Coefficient of thermal expansion, α (in/in/$^\circ$F)
80	6.47×10^{-6}
40	6.24×10^{-6}
-40	5.72×10^{-6}
-120	5.09×10^{-6}
-200	4.30×10^{-6}
-280	3.33×10^{-6}
-340	2.45×10^{-6}

(a) Using a third order polynomial interpolant, find the change in the coefficient of thermal expansion at $T=80\,^\circ\text{F}$ and $T=-340\,^\circ\text{F}$.

(b) The data given in Table 3 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T=80\,^\circ\text{F}$ and $T=-340\,^\circ\text{F}$.

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(t) = a_0 + a_1T + a_2T^2 + a_3T^3$$

(a) Change in the thermal expansion coefficient at $80\,^\circ\text{F}$:

Since we want to find the rate of change in the thermal expansion coefficient at $T=80\,^\circ\text{F}$, and we are using a third order polynomial, we need to choose the four points closest to $T=80\,^\circ\text{F}$ that also bracket $T=80\,^\circ\text{F}$ to evaluate it.

The four points are $T_0 = 80\,^\circ\text{F}$, $T_1 = 40\,^\circ\text{F}$, $T_2 = -40\,^\circ\text{F}$ and $T_3 = -120\,^\circ\text{F}$.

$$T_0 = 80,\ \ \alpha(T_0) = 6.47\times10^{-6}$$
$$T_1 = 40,\ \ \alpha(T_1) = 6.24\times10^{-6}$$
$$T_2 = -40,\ \ \alpha(T_2) = 5.72\times10^{-6}$$
$$T_3 = -120,\ \ \alpha(T_3) = 5.09\times10^{-6}$$

such that

$$\alpha(80) = 6.47\times10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$
$$\alpha(40) = 6.24\times10^{-6} = a_0 + a_1(40) + a_2(40)^2 + a_3(40)^3$$
$$\alpha(-40) = 5.72\times10^{-6} = a_0 + a_1(-40) + a_2(-40)^2 + a_3(-40)^3$$

$$\alpha(-120) = 5.09\times10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 512000 \\ 1 & 40 & 1600 & 64000 \\ 1 & -40 & 1600 & -64000 \\ 1 & -120 & 14400 & -1728000 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47\times10^{-6} \\ 6.24\times10^{-6} \\ 5.72\times10^{-6} \\ 5.09\times10^{-6} \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.59915\times10^{-5}$$
$$a_1 = 0.64813\times10^{-8}$$
$$a_2 = -0.71875\times10^{-11}$$
$$a_3 = 0.11719\times10^{-13}$$

Hence

$$\begin{aligned}\alpha(T) &= a_0 + a_1T + a_2T^2 + a_3T^3 \\ &= 0.59915\times10^{-5} + 0.64813\times10^{-8}T - 0.71875\times10^{-11}T^2 \\ &\quad + 0.11719\times10^{-13}T^3, \quad -120 \le T \le 80\end{aligned}$$

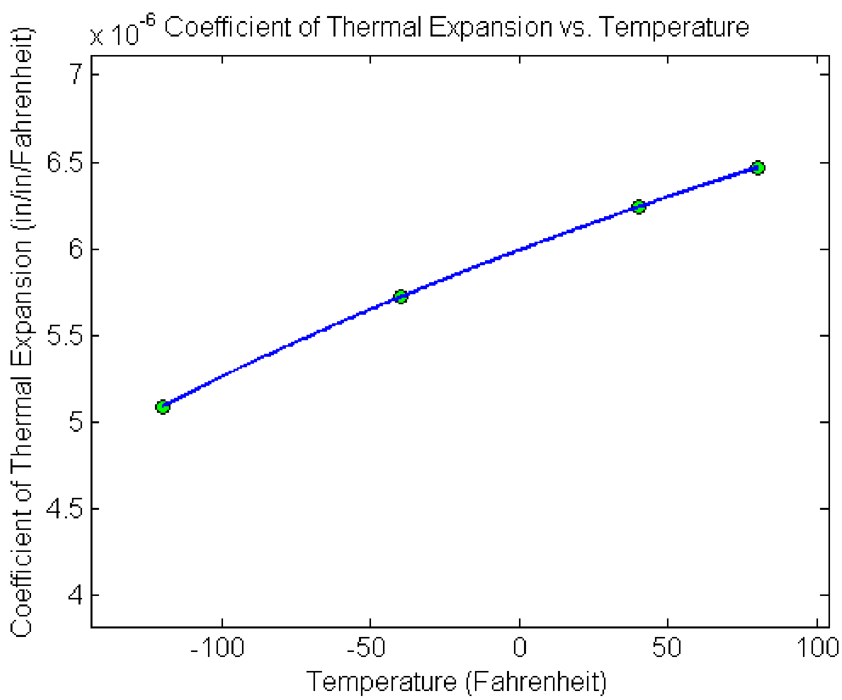

Figure 1 Graph of coefficient of thermal expansion vs. temperature.

The change in the coefficient of thermal expansion at $T = 80\ ^\circ\mathrm{F}$ is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=80}$$

Given that

$$\alpha(T) = 0.59915\times10^{-5} + 0.64813\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3, \quad -120 \le T \le 80$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT}\alpha(T)$$

$$= \frac{d}{dT}\begin{pmatrix} 0.59915\times10^{-5} + 0.64813\times10^{-8}T - 0.71875\times10^{-11}T^2 \\ + 0.11719\times10^{-13}T^3 \end{pmatrix}$$

$$= 0.64813\times10^{-8} - 1.4375\times10^{-11}T + 0.35157\times10^{-13}T^2, \quad -120 \le T \le 80$$

$$\frac{d\alpha(80)}{dT} = 0.64812\times10^{-8} - 1.4375\times10^{-11}(80) + 0.35157\times10^{-13}(80)^2$$

$$= 5.5563\times10^{-9} \text{ in/in/°F}^2$$

(b) Change in thermal expansion coefficient at $-340\ °\text{F}$:

Since we want to find the rate of change in the thermal expansion coefficient at $T = -340\ °\text{F}$, and we are using a third order polynomial, we need to choose the four points closest to $T = -340\ °\text{F}$ that also bracket $T = -340\ °\text{F}$ to evaluate it.

The four points are $T_0 = -120\ °\text{F},\ T_1 = -200\ °\text{F},\ T_2 = -280\ °\text{F}$ and $T_3 = -340\ °\text{F}$.

$$T_0 = -120,\ \ \alpha(T_0) = 5.09\times10^{-6}$$
$$T_1 = -200,\ \ \alpha(T_1) = 4.30\times10^{-6}$$
$$T_2 = -280,\ \ \alpha(T_2) = 3.33\times10^{-6}$$
$$T_3 = -340,\ \ \alpha(T_3) = 2.45\times10^{-6}$$

such that

$$\alpha(-120) = 5.09\times10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$
$$\alpha(-200) = 4.30\times10^{-6} = a_0 + a_1(-200) + a_2(-200)^2 + a_3(-200)^3$$
$$\alpha(-280) = 3.33\times10^{-6} = a_0 + a_1(-280) + a_2(-280)^2 + a_3(-280)^3$$
$$\alpha(-340) = 2.45\times10^{-6} = a_0 + a_1(-340) + a_2(-340)^2 + a_3(-340)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & -120 & 14400 & -1728000 \\ 1 & -200 & 40000 & -8000000 \\ 1 & -280 & 78400 & -21952000 \\ 1 & -340 & 115600 & -39304000 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.09\times10^{-6} \\ 4.30\times10^{-6} \\ 3.33\times10^{-6} \\ 2.45\times10^{-6} \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.60625\times10^{-5}$$
$$a_1 = 0.74881\times10^{-8}$$
$$a_2 = -0.29018\times10^{-11}$$
$$a_3 = 0.18601\times10^{-13}$$

Hence

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

$$= 0.60625\times10^{-5} + 0.74881\times10^{-8}T - 0.29018\times10^{-11}T^2$$
$$+ 0.18601\times10^{-13}T^3, \quad -340 \le T \le -120$$

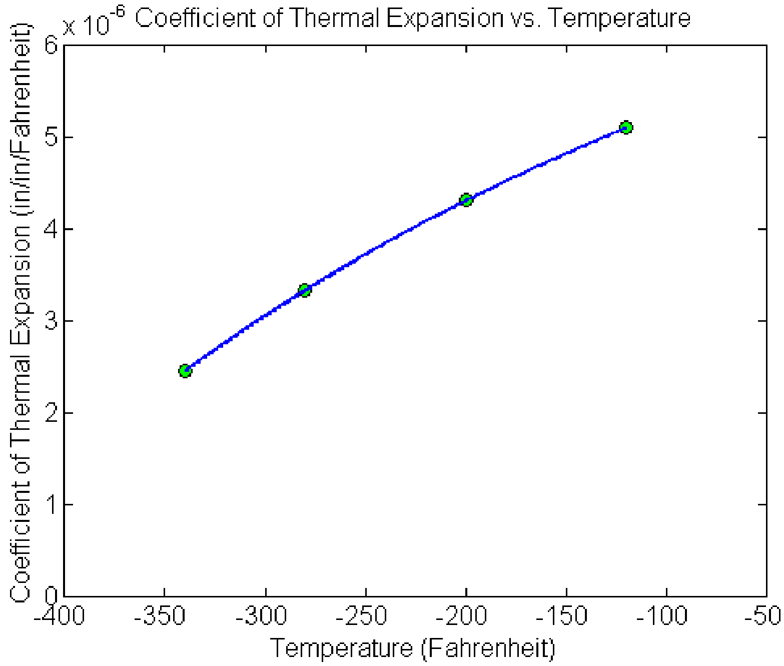

Figure 2 Graph of coefficient of thermal expansion vs. temperature.

The change in the coefficient of thermal expansion at $T = -340\ ^\circ\text{F}$ is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=-340}$$

Given that

$$\alpha(T) = 0.60625\times10^{-5} + 0.74881\times10^{-8}T - 0.29018\times10^{-11}T^2$$
$$+ 0.18601\times10^{-13}T^3, \quad -340 \le T \le -120$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT}\alpha(T)$$
$$= \frac{d}{dT}\begin{pmatrix} 0.60625\times10^{-5} + 0.74881\times10^{-8}T - 0.29018\times10^{-11}T^2 \\ + 0.18601\times10^{-13}T^3 \end{pmatrix}$$
$$= 0.74881\times10^{-8} - 0.58036\times10^{-11}T + 0.55804\times10^{-13}T^2, \quad -340 \le t \le -120$$

$$\frac{d\alpha(-340)}{dT} = 0.74881\times10^{-8} - 0.58036\times10^{-11}(-340) + 0.55804\times10^{-13}(-340)^2$$
$$= 0.15905\times10^{-7}\ \text{in/in/}^\circ\text{F}^2$$

Table 4 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	3[rd] Order Interpolation	2[nd] Order Polynomial Regression
80	5.5563×10^{-9} in/in/°F^2	4.3246×10^{-9} in/in/°F^2
-340	0.15905×10^{-7} in/in/°F^2	0.14585×10^{-7} in/in/°F^2

Example 3

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 5.

Table 5 Coefficient of thermal expansion as a function of temperature.

Temperature, T (°F)	Coefficient of thermal expansion, α (in/in/°F)
80	6.47×10^{-6}
40	6.24×10^{-6}
-40	5.72×10^{-6}
-120	5.09×10^{-6}
-200	4.30×10^{-6}
-280	3.33×10^{-6}
-340	2.45×10^{-6}

(a) Using a second order Lagrange polynomial interpolant, find the change in the coefficient of thermal expansion at $T = 80\ °\text{F}$ and $T = -340\ °\text{F}$.

(b) The data given in the Table 5 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80\ °\text{F}$ and $T = -340\ °\text{F}$.

Solution

For second order Lagrangian interpolation, we choose the coefficient of thermal expansion given by

$$\alpha(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\alpha(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\alpha(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\alpha(T_2)$$

(a) Change in the thermal expansion coefficient at $80\ °\text{F}$:

Since we want to find the rate of change in the thermal expansion coefficient at $T = 80\ °\text{F}$, and we are using second order Lagrangian interpolation, we need to choose the three points closest to $T = 80\ °\text{F}$ that also bracket $T = 80\ °\text{F}$ to evaluate it.

The three points are $T_0 = 80\ °\text{F}$, $T_1 = 40\ °\text{F}$ and $T_2 = -40\ °\text{F}$.

$$T_0 = 80,\ \ \alpha(T_0) = 6.47\times10^{-6}$$
$$T_1 = 40,\ \ \alpha(T_1) = 6.24\times10^{-6}$$
$$T_2 = -40,\ \ \alpha(T_2) = 5.72\times10^{-6}$$

The change in the coefficient of thermal expansion at $T = 80\,°\mathrm{F}$ is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=80}$$

Hence

$$\frac{d\alpha}{dT}(T) = \frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0) + \frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1) + \frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\frac{d\alpha}{dT}(80) = \frac{2(80)-(40+(-40))}{(80-40)(80-(-40))}\left(6.47\times10^{-6}\right) + \frac{2(80)-(80+(-40))}{(40-80)(40-(-40))}\left(6.24\times10^{-6}\right)$$
$$+\frac{2(80)-(80+40)}{(-40-80)(-40-40)}\left(5.72\times10^{-6}\right)$$
$$= 2.1567\times10^{-7} - 2.34\times10^{-7} + 2.3833\times10^{-8}$$
$$= 5.5\times10^{-9}\ \text{in/in/°F}^2$$

(b) Change in the thermal expansion coefficient at $-340\,°\mathrm{F}$:

Since we want to find the rate of change in the thermal expansion coefficient at $T = -340\,°\mathrm{F}$, and we are using second order Lagrangian interpolation, we need to choose the three points closest to $T = -340\,°\mathrm{F}$ that also bracket $T = -340\,°\mathrm{F}$ to evaluate it.

The three points are $T_0 = -200\,°\mathrm{F}$, $T_1 = -280\,°\mathrm{F}$ and $T_2 = -340\,°\mathrm{F}$.

$$T_0 = -200,\ \ \alpha(T_0) = 4.30\times10^{-6}$$
$$T_1 = -280,\ \ \alpha(T_1) = 3.33\times10^{-6}$$
$$T_2 = -340,\ \ \alpha(T_2) = 2.45\times10^{-6}$$

The change in the coefficient of thermal expansion at $T = -340\,°\mathrm{F}$ is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=-340}$$

Hence

$$\frac{d\alpha}{dT}(T) = \frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0) + \frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1) + \frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\frac{d\alpha}{dT}(-340) = \frac{2(-340)-(-280+(-340))}{(200-(-280))(-200-(-340))}\left(4.30\times10^{-6}\right)$$
$$+\frac{2(-340)-(-200+(-280))}{(-280-(-200))(-280-(-340))}\left(3.33\times10^{-6}\right)$$
$$+\frac{2(-340)-(-200+(-280))}{(-340-(-200))(-340-(-280))}\left(2.45\times10^{-6}\right)$$
$$= -2.3036\times10^{-8} + 9.7125\times10^{-8} - 5.8333\times10^{-8}$$
$$= 0.15756\times10^{-7}\ \text{in/in/°F}^2$$

Table 6 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	2nd Order Lagrange Interpolation	2nd Order Polynomial Regression
80	5.5×10^{-9} in/in/°F^2	4.3246×10^{-9} in/in/°F^2
-340	0.15756×10^{-7} in/in/°F^2	0.14585×10^{-7} in/in/°F^2

DIFFERENTIATION	
Topic	Discrete Functions-More Examples
Summary	Examples of Discrete Functions
Major	Mechanical Engineering
Authors	Autar Kaw
Date	August 7, 2009
Web Site	http://numericalmethods.eng.usf.edu