Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Industrial Engineering

Example 1

The open loop response, that is, the speed of the motor to a voltage input of 20V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w.$$

If the initial speed is zero $(w(0) = 0)$,and using the Runge-Kutta 4th order method, what is the speed at $t = 0.8\,\text{s}$? Assume a step size of $h = 0.4\,\text{s}$.

Solution

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t,w) = 1000 - 3w$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $w_0 = 0$

$$\begin{aligned}
k_1 &= f(t_0, w_0) \\
&= f(0,0) \\
&= 1000 - 3 \times 0 \\
&= 1000
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \left(\frac{1}{2} \times 0.4\right), 0 + \left(\frac{1}{2}(1000) \times 0.4\right)\right) \\
&= f(0.2,200) \\
&= 1000 - 3 \times 200 \\
&= 400
\end{aligned}$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0 + \left(\frac{1}{2} \times 0.4\right), 0 + \left(\frac{1}{2}(400) \times 0.4\right)\right)$$
$$= f(0.2, 80)$$
$$= 1000 - 3 \times 80$$
$$= 760$$

$$k_4 = f(t_0 + h, w_0 + k_3 h)$$
$$= f(0 + (0.4), 0 + ((760) \times 0.4))$$
$$= f(0.4, 304)$$
$$= 1000 - 3 \times 304$$
$$= 88$$

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 0 + \frac{1}{6}(1000 + 2 \times (400) + 2 \times (760) + (88)) \times 0.4$$
$$= 0 + \frac{1}{6}(3408) \times 0.4$$
$$= 227.2 \text{ rad/s}$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4\,\text{s}$$
$$w(0.4) \approx w_1 = 227.2 \text{ rad/s}$$

For $i = 1,\ t_1 = 0.4,\ w_1 = 227.2$

$$k_1 = f(t_1, w_1)$$
$$= f(0.4, 227.2)$$
$$= 1000 - 3 \times 227.2$$
$$= 318.4$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, w_1 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0.4 + \left(\frac{1}{2} \times 0.4\right), 227.2 + \left(\frac{1}{2}(318.4) \times 0.4\right)\right)$$
$$= f(0.6, 290.88)$$
$$= 1000 - 3 \times 290.88$$
$$= 127.36$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, w_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0.4 + \left(\frac{1}{2} \times 0.4\right), 227.2 + \left(\frac{1}{2}(127.36) \times 0.4\right)\right)$$
$$= f(0.6, 252.67)$$
$$= 1000 - 3 \times 252.67$$
$$= 241.98$$

$$
\begin{aligned}
k_4 &= f(t_1 + h, w_1 + k_3 h) \\
&= f(0.4 + 0.4, 227.2 + (241.98 \times 0.4)) \\
&= f(0.8, 323.99) \\
&= 1000 - 3 \times 323.99 \\
&= 28.019
\end{aligned}
$$

$$
\begin{aligned}
w_2 &= w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 227.2 + \frac{1}{6}(318.4 + 2 \times (127.36) + 2 \times (241.98) + 28.019) \times 0.4 \\
&= 227.2 + \frac{1}{6}(1085.1) \times 0.4 \\
&= 299.54 \text{ rad/s}
\end{aligned}
$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\, s$$

$$w(0.8) \approx w_2 = 299.54 \text{ rad/s}$$

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right) e^{-3t}$$

The solution to this nonlinear equation at $t = 0.8\text{s}$ is

$$w(0.8) = 303.09 \text{ rad/s}$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

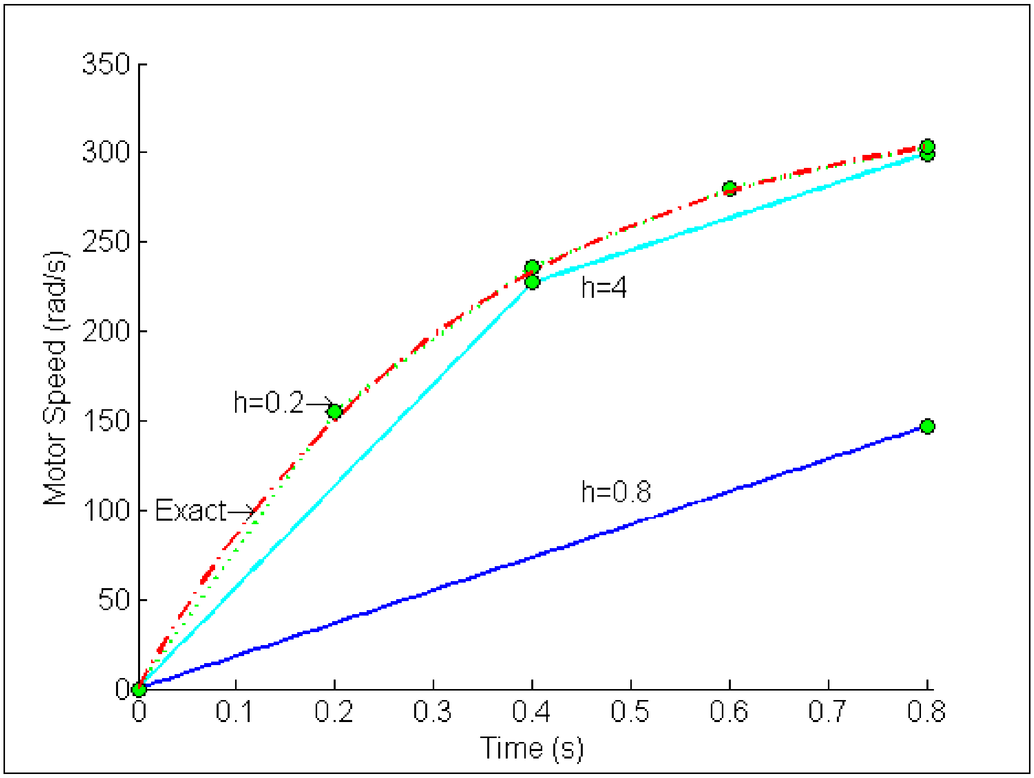

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated speed of the motor at $t = 0.8\,\text{s}$.

Table 1 Values of speed of the motor at 0.8 seconds for different step sizes.

Step size, h	$w(0.8)$	E_t	$\|\in_t\|\%$
0.8	147.20	155.89	51.434
0.4	299.54	3.5535	1.1724
0.2	302.96	0.12988	0.042852
0.1	303.09	0.0062962	0.0020773
0.05	303.09	0.00034702	0.00011449

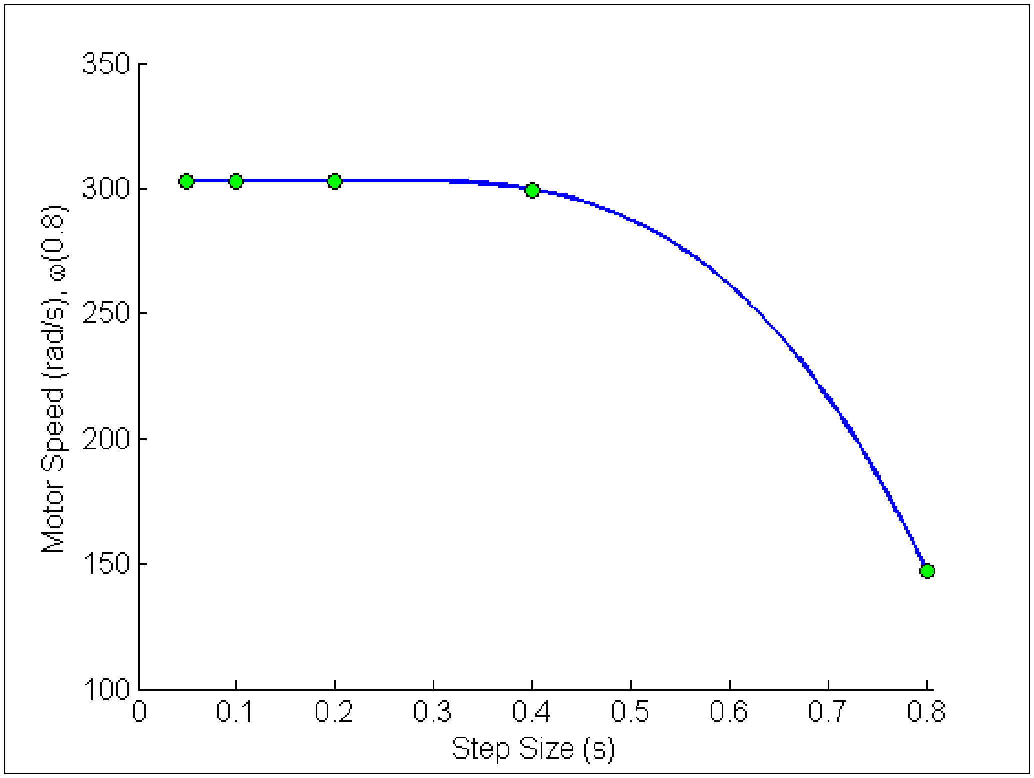

Figure 2 Effect of step size in Runge-Kutta 4^{th} order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1^{st} order method), Heun's method (Runge-Kutta 2^{nd} order method) and the Runge-Kutta 4^{th} order method.

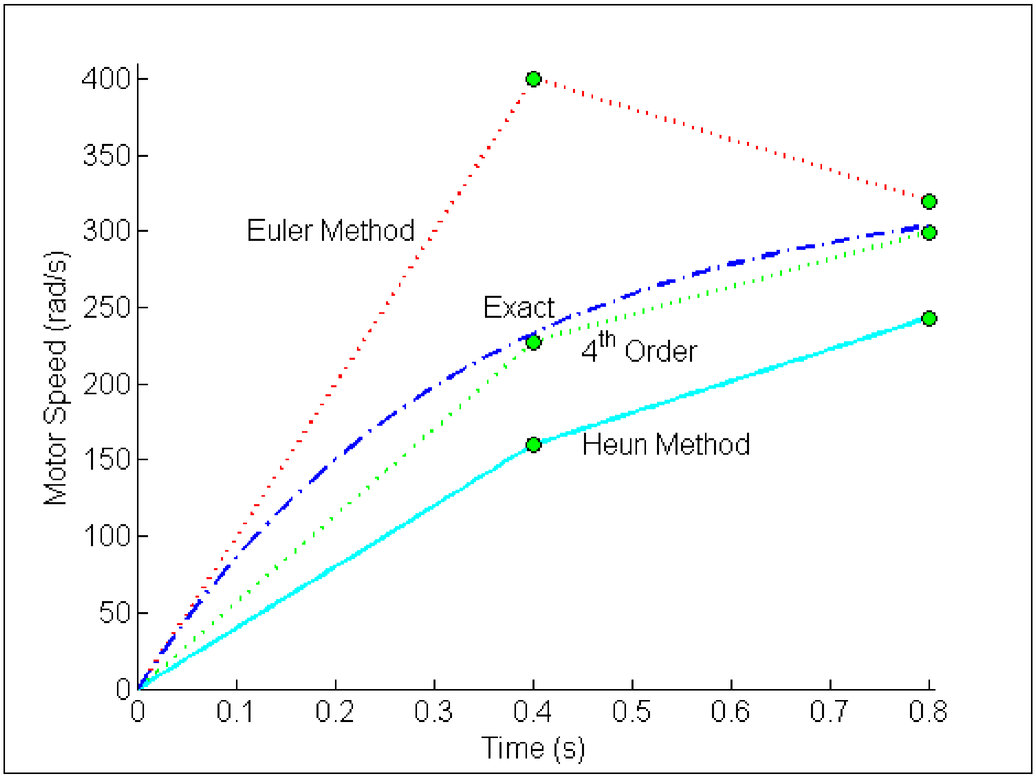

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.