Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Industrial Engineering

Example 1

The open loop response, that is, the speed of the motor to a voltage input of 20V, assuming a system without damping is

\[ 20 = (0.02) \frac{dw}{dt} + (0.06)w. \]

If the initial speed is zero \((w(0) = 0)\), and using the Runge-Kutta 4\(^{th}\) order method, what is the speed at \(t = 0.8\) s? Assume a step size of \(h = 0.4\) s.

Solution

\[
\frac{dw}{dt} = 1000 - 3w
\]

\[ f(t, w) = 1000 - 3w \]

\[ w_{i+1} = w_i \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right)h \]

For \(i = 0\), \(t_0 = 0\), \(w_0 = 0\)

\[ k_1 = f(t_0, w_0) \]

\[ = f(0, 0) \]

\[ = 1000 - 3 \times 0 \]

\[ = 1000 \]

\[ k_2 = f\left( t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_1h \right) \]

\[ = f\left( 0 + \frac{1}{2} \times 0.4, 0 + \frac{1}{2} (1000) \times 0.4 \right) \]

\[ = f(0.2, 200) \]

\[ = 1000 - 3 \times 200 \]

\[ = 400 \]

\[ k_3 = f\left( t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_2h \right) \]

\[ = f\left( 0 + \frac{1}{2} \times 0.4, 0 + \frac{1}{2} (200) \times 0.4 \right) \]

\[ = f(0.2, 100) \]

\[ = 1000 - 3 \times 100 \]

\[ = 400 \]

\[ k_4 = f\left( t_0 + h, w_0 + k_3h \right) \]

\[ = f\left( 0 + 0.4, 0 + 400 \right) \]

\[ = f(0.4, 400) \]

\[ = 1000 - 3 \times 400 \]

\[ = -600 \]

\[ w_1 = w_0 + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right)h \]

\[ = 0 + \frac{1}{6} \left( 1000 + 2 \times 400 + 2 \times 400 - 600 \right) \]

\[ = 0 + \frac{1}{6} \left( 1000 + 800 + 800 - 600 \right) \]

\[ = 0 + \frac{1}{6} \times 1200 \]

\[ = 200 \]
\[ f(t) = 0 + \left( \frac{1}{2} \times 0.4 \right), 0 + \left( \frac{1}{2} \times (400) \times 0.4 \right) \]
\[ f(0.2, 80) \]
\[ 1000 - 3 \times 80 \]
\[ 760 \]
\[ k_4 = f(t_0 + h, w_0 + k_3 h) \]
\[ f(0 + (0.4), 0 + ((760) \times 0.4)) \]
\[ f(0.4, 304) \]
\[ 1000 - 3 \times 304 \]
\[ 88 \]
\[ w_i = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \]
\[ 0 + \frac{1}{6}(1000 + 2 \times (400) + 2 \times (760) + (88)) \times 0.4 \]
\[ 0 + \frac{1}{6}(3408) \times 0.4 \]
\[ 227.2 \text{ rad/s} \]

\( w_1 \) is the approximate speed of the motor at
\( t = t_i = t_0 + h = 0 + 0.4 = 0.4 \text{ s} \)
\[ w(0.4) \approx w_1 = 227.2 \text{ rad/s} \]

For \( i = 1, \ t_i = 0.4, \ w_i = 227.2 \)
\[ k_1 = f(t_i, w_i) \]
\[ f(0.4, 227.2) \]
\[ 1000 - 3 \times 227.2 \]
\[ 318.4 \]
\[ k_2 = f(t_i + \frac{1}{2} h, w_i + \frac{1}{2} k_i h) \]
\[ f(0.4 + \left( \frac{1}{2} \times 0.4 \right), 227.2 + \left( \frac{1}{2} \times (318.4) \times 0.4 \right)) \]
\[ f(0.6, 290.88) \]
\[ 1000 - 3 \times 290.88 \]
\[ 127.36 \]
\[ k_3 = f(t_i + \frac{1}{2} h, w_i + \frac{1}{2} k_i h) \]
\[ f(0.4 + \left( \frac{1}{2} \times 0.4 \right), 227.2 + \left( \frac{1}{2} \times (127.36) \times 0.4 \right)) \]
\[ f(0.6, 252.67) \]
\[ 1000 - 3 \times 252.67 \]
\[ 241.98 \]


\[ k_4 = f(t_1 + h, w_i + k_3 h) \]
\[ = f(0.4 + 0.4, 227.2 + (241.98 \times 0.4)) \]
\[ = f(0.8, 323.99) \]
\[ = 1000 - 3 \times 323.99 \]
\[ = 28.019 \]

\[ w_2 = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h \]
\[ = 227.2 + \frac{1}{6} (318.4 + 2 \times (127.36) + 2 \times (241.98) + 28.019) \times 0.4 \]
\[ = 227.2 + \frac{1}{6} (1085.1) \times 0.4 \]
\[ = 299.54 \text{ rad/s} \]

\( w_2 \) is the approximate speed of the motor at
\[ t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8 \text{ s} \]
\[ w(0.8) \approx w_2 = 299.54 \text{ rad/s} \]

The exact solution of the ordinary differential equation is given by
\[ w(t) = \left( \frac{1000}{3} \right) - \left( \frac{1000}{3} \right) e^{-3t} \]

The solution to this nonlinear equation at \( t = 0.8 \text{ s} \) is
\[ w(0.8) = 303.09 \text{ rad/s} \]

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.
Figure 1 Comparison of Runge-Kutta 4\textsuperscript{th} order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated speed of the motor at $t = 0.8\text{ s}$.

Table 1 Values of speed of the motor at 0.8 seconds for different step sizes.

| Step size, $h$ | $w(0.8)$ | $E_i$     | $|\varepsilon_i|\ %$ |
|---------------|-----------|-----------|-----------------|
| 0.8           | 147.20    | 155.89    | 51.434          |
| 0.4           | 299.54    | 3.5535    | 1.1724          |
| 0.2           | 302.96    | 0.12988   | 0.042852        |
| 0.1           | 303.09    | 0.0062962 | 0.0020773       |
| 0.05          | 303.09    | 0.00034702| 0.00011449      |
In Figure 2, we are comparing the exact results with Euler’s method (Runge-Kutta 1$^{st}$ order method), Heun’s method (Runge-Kutta 2$^{nd}$ order method) and the Runge-Kutta 4$^{th}$ order method.

In Figure 3, we are comparing the exact results with Euler’s method (Runge-Kutta 1$^{st}$ order method), Heun’s method (Runge-Kutta 2$^{nd}$ order method) and the Runge-Kutta 4$^{th}$ order method.
Figure 3  Comparison of Runge-Kutta methods of 1\textsuperscript{st}, 2\textsuperscript{nd}, and 4\textsuperscript{th} order.