Chapter 05.05
Spline Method of Interpolation – More Examples
Industrial Engineering

Example 1

The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

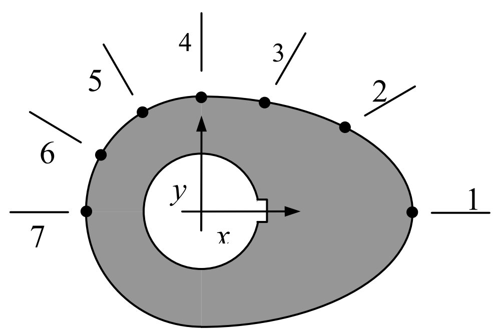

Figure 1 Schematic of cam profile.

Table 1 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

a) If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using linear splines.
b) Find the cam profile using linear splines.

Solution

a) Since we want to find the value of y at $x = 1.10$, and we are using linear splines, we need to choose the two data points that are closest to $x = 1.10$ that also bracket $x = 1.10$ to evaluate it. The two point are $x_0 = 1.28$ and $x_1 = 0.66$.

Then

$$x_0 = 1.28, \quad y(x_0) = 0.88$$
$$x_1 = 0.66, \quad y(x_1) = 1.14$$

gives

$$\begin{aligned} y(x) &= y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0) \\ &= 0.88 + \frac{1.14 - 0.88}{0.66 - 1.28}(x - 1.28) \end{aligned}$$

Hence

$$y(x) = 0.88 - 0.41935(x - 1.28), \ 0.66 \le x \le 1.28$$

At $x = 1.10$,

$$\begin{aligned} y(1.10) &= 0.88 - 0.41935(1.10 - 1.28) \\ &= 0.95548 \text{ in.} \end{aligned}$$

b) Going in increasing order of x-values, the first linear spline connects $x = -1.20$ and $x = -1.04$.

$$\begin{aligned} y(x) &= 0.00 + \frac{0.60 - 0.00}{-1.04 + 1.20}(x + 1.20) \\ &= 3.75(x + 1.20), \qquad -1.20 \le x \le -1.04 \qquad (1) \end{aligned}$$

The rest of the splines are as follows.

$$\begin{aligned} y(x) &= 0.60 + \frac{1.04 - 0.60}{-0.60 + 1.04}(x + 1.04) \\ &= x + 1.64, \qquad -1.04 \le x \le -0.60 \qquad (2) \end{aligned}$$

$$\begin{aligned} y(x) &= 1.04 + \frac{1.20 - 1.04}{0.00 + 0.60}(x + 0.60) \\ &= 1.04 + 0.26667(x + 0.60), \qquad -0.60 \le x \le 0.00 \qquad (3) \end{aligned}$$

$$\begin{aligned} y(x) &= 1.20 + \frac{1.14 - 1.20}{0.66 - 0.00}(x - 0.00) \\ &= 1.20 - 0.090909x, \qquad 0.00 \le x \le 0.66 \qquad (4) \end{aligned}$$

$$\begin{aligned} y(x) &= 1.14 + \frac{0.88 - 1.14}{1.28 - 0.66}(x - 0.66) \\ &= 1.14 - 0.41935(x - 0.66), \qquad 0.66 \le x \le 1.28 \qquad (5) \end{aligned}$$

$$\begin{aligned} y(x) &= 0.88 + \frac{0.00 - 0.88}{2.20 - 1.28}(x - 1.28) \\ &= 0.88 - 0.95652(x - 1.28), \qquad 1.28 \le x \le 2.20 \qquad (6) \end{aligned}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

The geometry of a cam is given in Figure 2. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

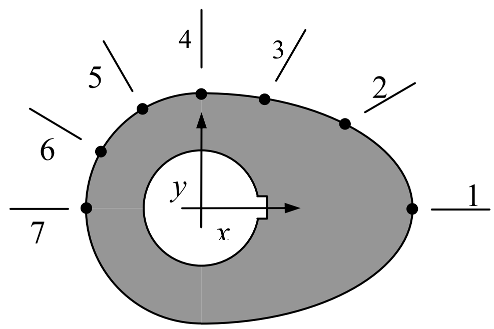

Figure 2 Schematic of cam profile.

Table 2 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

a) Find the cam profile using quadratic splines.
b) Compare the profile with the linear spline and a sixth order polynomial result.

Solution

a) Since there are seven data points, six quadratic splines pass through them.

$$\begin{aligned}
y(x) &= a_1x^2 + b_1x + c_1, && -1.20 \le x \le -1.04 \\
&= a_2x^2 + b_2x + c_2, && -1.04 \le x \le -0.60 \\
&= a_3x^2 + b_3x + c_3, && -0.60 \le x \le 0.00 \\
&= a_4x^2 + b_4x + c_4, && 0.00 \le x \le 0.66 \\
&= a_5x^2 + b_5x + c_5, && 0.66 \le x \le 1.28
\end{aligned}$$

$$= a_6x^2 + b_6x + c_6, \qquad 1.28 \le x \le 2.20$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1x^2 + b_1x + c_1$ passes through $x = -1.20$ and $x = -1.04$.

$$a_1(-1.20)^2 + b_1(-1.20) + c_1 = 0.00 \qquad (1)$$

$$a_1(-1.04)^2 + b_1(-1.04) + c_1 = 0.60 \qquad (2)$$

$a_2x^2 + b_2x + c_2$ passes through $x = -1.04$ and $x = -0.60$.

$$a_2(-1.04)^2 + b_2(-1.04) + c_2 = 0.60 \qquad (3)$$

$$a_2(-0.60)^2 + b_2(-0.60) + c_2 = 1.04 \qquad (4)$$

$a_3x^2 + b_3x + c_3$ passes through $x = -0.60$ and $x = 0.00$.

$$a_3(-0.60)^2 + b_3(-0.60) + c_3 = 1.04 \qquad (5)$$

$$a_3(0.00)^2 + b_3(0.00) + c_3 = 1.20 \qquad (6)$$

$a_4x^2 + b_4x + c_4$ passes through $x = 0.00$ and $x = 0.66$.

$$a_4(0.00)^2 + b_4(0.00) + c_4 = 1.20 \qquad (7)$$

$$a_4(0.66)^2 + b_4(0.66) + c_4 = 1.14 \qquad (8)$$

$a_5x^2 + b_5x + c_5$ passes through $x = 0.66$ and $x = 1.28$.

$$a_5(0.66)^2 + b_5(0.66) + c_5 = 1.14 \qquad (9)$$

$$a_5(1.28)^2 + b_5(1.28) + c_5 = 0.88 \qquad (10)$$

$a_6x^2 + b_6x + c_6$ passes through $x = 1.28$ and $x = 2.20$.

$$a_6(1.28)^2 + b_6(1.28) + c_6 = 0.88 \qquad (11)$$

$$a_6(2.20)^2 + b_6(2.20) + c_6 = 0.00 \qquad (12)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $x = -1.04$

$$2a_1(-1.04) + b_1 - 2a_2(-1.04) - b_2 = 0 \qquad (13)$$

At $x = -0.60$

$$2a_2(-0.60) + b_2 - 2a_3(-0.60) - b_3 = 0 \qquad (14)$$

At $x = 0.00$

$$2a_3(0.00) + b_3 - 2a_4(0.00) - b_4 = 0 \qquad (15)$$

At $x = 0.66$

$$2a_4(0.66) + b_4 - 2a_5(0.66) - b_5 = 0 \qquad (16)$$

At $x = 1.28$

$$2a_5(1.28) + b_5 - 2a_6(1.28) - b_6 = 0 \qquad (17)$$

3. Assuming the first spline $a_1x^2 + b_1x + c_1$ is linear,

$$a_1 = 0 \tag{18}$$

$$\begin{bmatrix}
1.44 & -1.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0816 & -1.04 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0816 & -1.04 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.36 & -0.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.36 & -0.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4356 & 0.66 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4356 & 0.66 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6384 & 1.28 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6384 & 1.28 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.84 & 2.2 & 1 \\
-2.08 & 1 & 0 & 2.08 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.2 & 1 & 0 & 1.2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.32 & 1 & 0 & -1.32 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.56 & 1 & 0 & -2.56 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \\ a_6 \\ b_6 \\ c_6 \end{bmatrix}
=
\begin{bmatrix} 0.00 \\ 0.60 \\ 0.60 \\ 1.04 \\ 1.04 \\ 1.20 \\ 1.20 \\ 1.14 \\ 1.14 \\ 0.88 \\ 0.88 \\ 0.00 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 18 equations gives the 18 unknowns as

i	a_i	b_i	c_i
1	0	3.75	4.5
2	–6.25	–9.25	–2.26
3	3.3611	2.2833	1.2
4	–3.5973	2.2833	1.2
5	3.2997	–6.8207	4.2043
6	–2.8076	8.8138	–5.8018

Therefore, the splines are given by

$$\begin{aligned}
y(x) &= 3.75x + 4.5, & -1.20 \le x \le -1.04 \\
&= -6.25x^2 - 9.25x - 2.26, & -1.04 \le x \le -0.60 \\
&= 3.3611x^2 + 2.2833x + 1.2, & -0.60 \le x \le 0.00 \\
&= -3.5973x^2 + 2.2833x + 1.2, & 0.00 \le x \le 0.66 \\
&= 3.2997x^2 - 6.8207x + 4.2043, & 0.66 \le x \le 1.28 \\
&= -2.8076x^2 + 8.8138x - 5.8018, & 1.28 \le x \le 2.20
\end{aligned}$$

b)

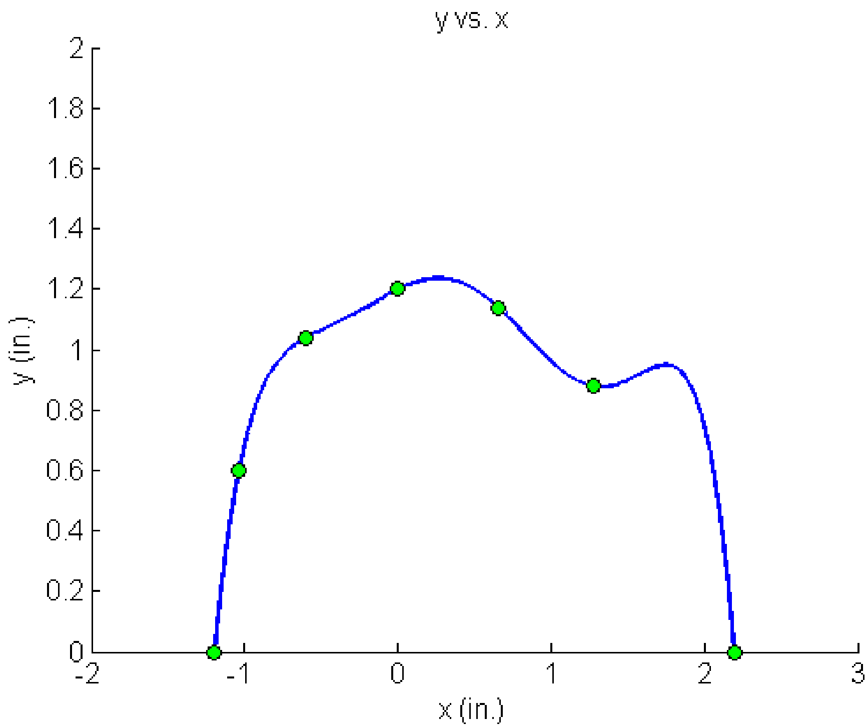

Figure 3 Plot of the cam profile as defined by linear splines, quadratic splines and a 6^{th} order polynomial.