Chapter 05.04
Lagrange Method of Interpolation – More Examples
Industrial Engineering

Example 1

The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

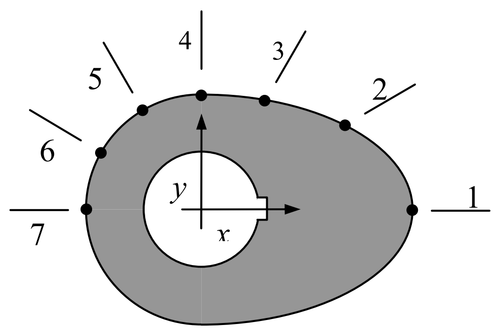

Figure 1 Schematic of cam profile.

Table 1 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using a first order Lagrange polynomial?

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the value of y is given by

$$y(x) = \sum_{i=0}^{1} L_i(x)y(x_i)$$
$$= L_0(x)y(x_0) + L_1(x)y(x_1)$$

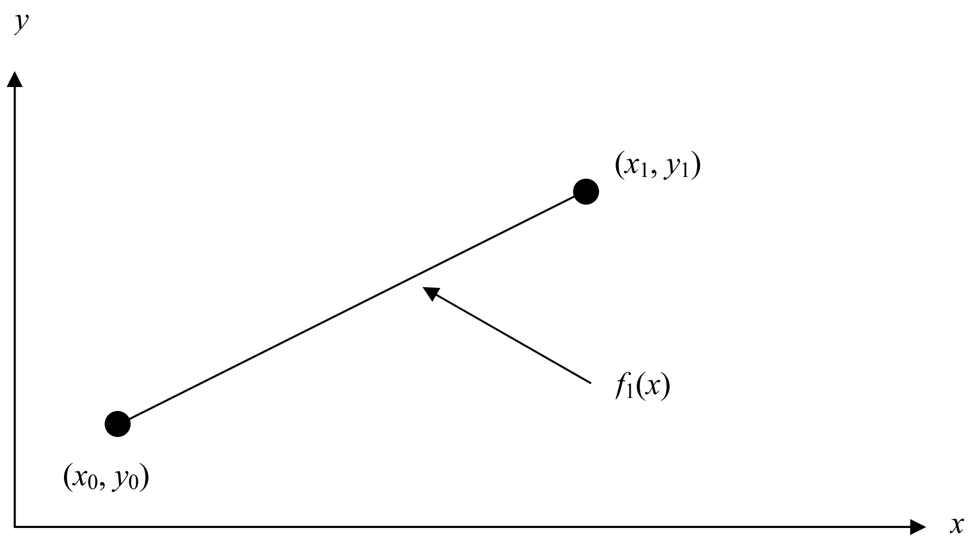

Figure 2 Linear interpolation.

Since we want to find the value of y at $x = 1.10$, using the two points $x_0 = 1.28$ and $x_1 = 0.66$, then

$$x_0 = 1.28,\ y(x_0) = 0.88$$
$$x_1 = 0.66,\ y(x_1) = 1.14$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_0 - x_j}$$
$$= \frac{x - x_1}{x_0 - x_1}$$
$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{x - x_j}{x_1 - x_j}$$
$$= \frac{x - x_0}{x_1 - x_0}$$

Hence

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1)$$

$$= \frac{x - 0.66}{1.28 - 0.66}(0.88) + \frac{x - 1.28}{0.66 - 1.28}(1.14), \quad 0.66 \le x \le 1.28$$

$$y(1.10) = \frac{1.10 - 0.66}{1.28 - 0.66}(0.88) + \frac{1.10 - 1.28}{0.66 - 1.28}(1.14)$$

$$= 0.70968(0.88) + 0.29032(1.14)$$

$$= 0.95548 \text{ in.}$$

You can see that $L_0(x) = 0.70968$ and $L_1(x) = 0.29032$ are like weightages given to the values of y at $x = 1.28$ and $x = 0.66$ to calculate the value of y at $x = 1.10$.

Example 2

The geometry of a cam is given in Figure 3. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

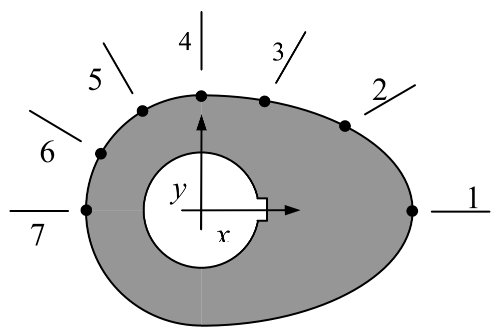

Figure 3 Schematic of cam profile.

Table 2 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a quadratic profile from $x = 2.20$ to $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using a second order Lagrange polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the value of y given by

$$y(x) = \sum_{i=0}^{2} L_i(x) y(x_i)$$
$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2)$$

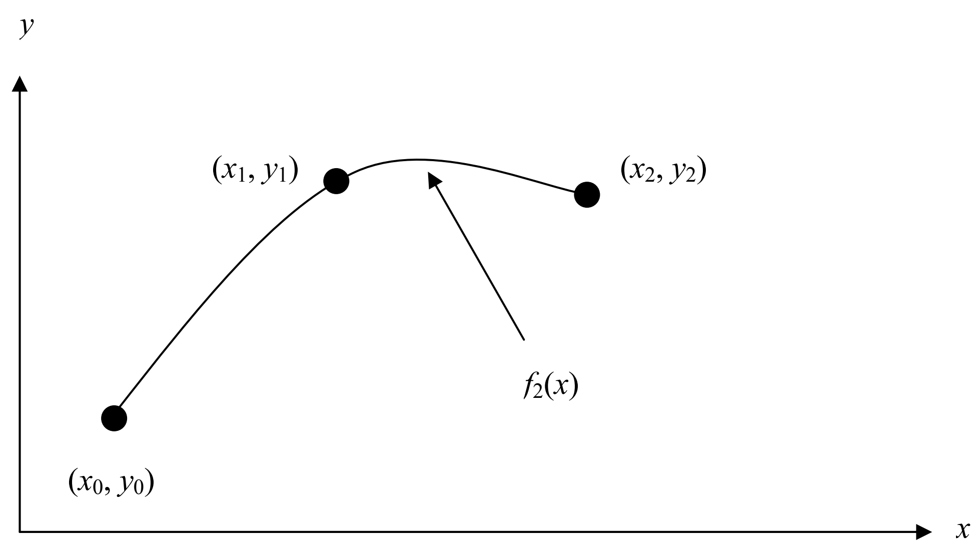

Figure 4 Quadratic interpolation.

Since we want to find the value of y at $x = 1.10$, using the three points $x_0 = 2.20$, $x_1 = 1.28$, $x_2 = 0.66$, then

$$x_0 = 2.20, \quad y(x_0) = 0.00$$
$$x_1 = 1.28, \quad y(x_1) = 0.88$$
$$x_2 = 0.66, \quad y(x_2) = 1.14$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x)=\prod_{\substack{j=0\\ j\neq 2}}^{2}\frac{x-x_j}{x_2-x_j}$$
$$=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)$$

Hence

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)y(x_0)+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)y(x_1)+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)y(x_2),$$
$$x_0\le x\le x_2$$

$$y(1.10)=\frac{(1.10-1.28)(1.10-0.66)}{(2.20-1.28)(2.20-0.66)}(0.00)+\frac{(1.10-2.20)(1.10-0.66)}{(1.28-2.20)(1.28-0.66)}(0.88)$$
$$+\frac{(1.10-2.20)(1.10-1.28)}{(0.66-2.20)(0.66-1.28)}(1.14)$$
$$=(-0.055901)(0.00)+(0.84853)(0.88)+(0.20737)(1.14)$$
$$=0.98311 \text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{0.98311-0.95548}{0.98311}\right|\times 100$$
$$=2.8100\%$$

Example 3

The geometry of a cam is given in Figure 5. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.

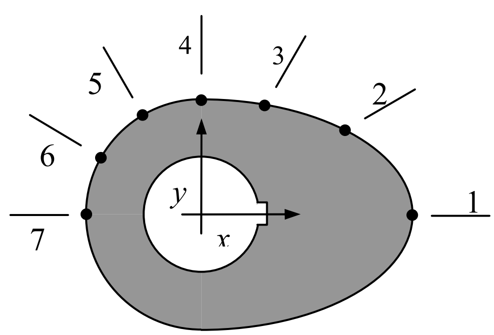

Figure 5 Schematic of cam profile.

Table 3 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

Find the cam profile using all seven points in Table 3 and a sixth order Lagrange polynomial.

Solution

For the sixth order polynomial, we choose the value of y given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{6} L_i(x)y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) \\ &\qquad + L_4(x)y(x_4) + L_5(x)y(x_5) + L_6(x)y(x_6) \end{aligned}$$

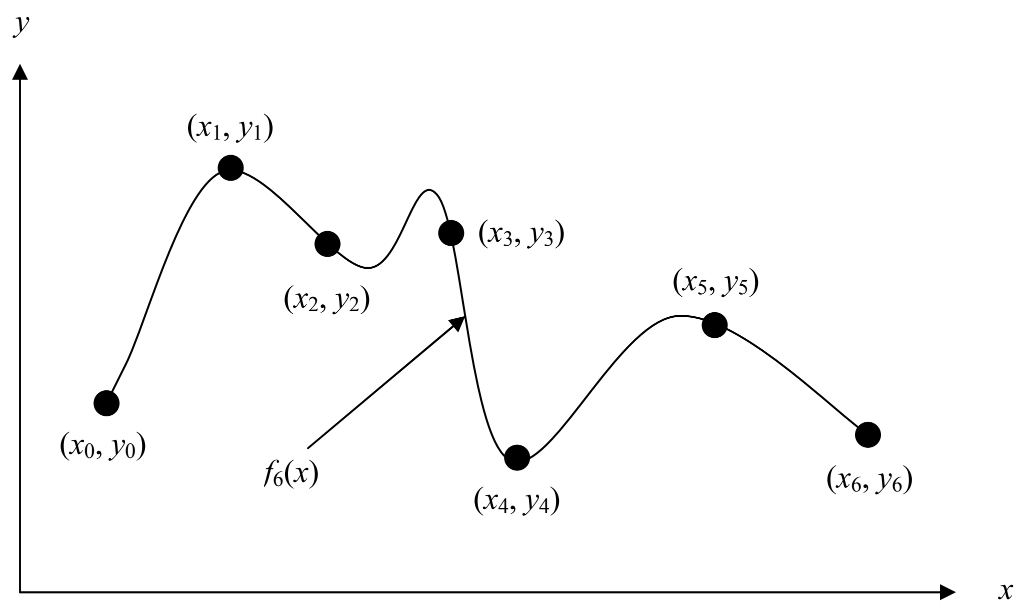

Figure 6 6th order polynomial interpolation.

Using the seven points,

$$x_0 = 2.20, \quad y(x_0) = 0.00$$
$$x_1 = 1.28, \quad y(x_1) = 0.88$$
$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$x_3 = 0.00, \quad y(x_3) = 1.20$$
$$x_4 = -.60, \quad y(x_4) = 1.04$$
$$x_5 = -1.04, \quad y(x_5) = 0.60$$
$$x_6 = -1.20, \quad y(x_6) = 0.00$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{6} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right)\left(\frac{x - x_4}{x_0 - x_4}\right)\left(\frac{x - x_5}{x_0 - x_5}\right)\left(\frac{x - x_6}{x_0 - x_6}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{6} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)\left(\frac{x - x_4}{x_1 - x_4}\right)\left(\frac{x - x_5}{x_1 - x_5}\right)\left(\frac{x - x_6}{x_1 - x_6}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{6} \frac{x - x_j}{x_2 - x_j}$$
$$= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right)\left(\frac{x - x_4}{x_2 - x_4}\right)\left(\frac{x - x_5}{x_2 - x_5}\right)\left(\frac{x - x_6}{x_2 - x_6}\right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j \neq 3}}^{6} \frac{x - x_j}{x_3 - x_j}$$
$$= \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)\left(\frac{x - x_4}{x_3 - x_4}\right)\left(\frac{x - x_5}{x_3 - x_5}\right)\left(\frac{x - x_6}{x_3 - x_6}\right)$$

$$L_4(x) = \prod_{\substack{j=0 \\ j \neq 4}}^{6} \frac{x - x_j}{x_4 - x_j}$$
$$= \left(\frac{x - x_0}{x_4 - x_0}\right)\left(\frac{x - x_1}{x_4 - x_1}\right)\left(\frac{x - x_2}{x_4 - x_2}\right)\left(\frac{x - x_3}{x_4 - x_3}\right)\left(\frac{x - x_5}{x_4 - x_5}\right)\left(\frac{x - x_6}{x_4 - x_6}\right)$$

$$L_5(x) = \prod_{\substack{j=0 \\ j \neq 5}}^{6} \frac{x - x_j}{x_5 - x_j}$$
$$= \left(\frac{x - x_0}{x_5 - x_0}\right)\left(\frac{x - x_1}{x_5 - x_1}\right)\left(\frac{x - x_2}{x_5 - x_2}\right)\left(\frac{x - x_3}{x_5 - x_3}\right)\left(\frac{x - x_4}{x_5 - x_4}\right)\left(\frac{x - x_6}{x_5 - x_6}\right)$$

$$L_6(x) = \prod_{\substack{j=0 \\ j \neq 6}}^{6} \frac{x - x_j}{x_6 - x_j}$$

$$= \left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)$$

$$\begin{aligned}
y(x) = &\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)y(x_0) \\
&+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)y(x_1) \\
&+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)y(x_2) \\
&+\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)y(x_3) \\
&+\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)y(x_4) \\
&+\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)y(x_5) \\
&+\left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)y(x_6)
\end{aligned}$$

$$\begin{aligned}
y(x) = &\frac{(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(2.20-1.28)(2.20-0.66)(2.20-0.00)(2.20+0.60)(2.20+1.04)(2.20+1.20)}(0.00) \\
&+\frac{(x-2.20)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(1.28-2.20)(1.28-0.66)(1.28-0.00)(1.28+0.60)(1.28+1.04)(1.28+1.20)}(0.88) \\
&+\frac{(x-2.20)(x-1.28)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(0.66-2.20)(0.66-1.28)(0.66-0.00)(0.66+0.60)(0.66+1.04)(0.66+1.20)}(1.14) \\
&+\frac{(x-2.20)(x-1.28)(x-0.66)(x+0.60)(x+1.04)(x+1.20)}{(0.00-2.20)(0.00-1.28)(0.00-0.66)(0.00+0.60)(0.00+1.04)(0.00+1.20)}(1.20) \\
&+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+1.04)(x+1.20)}{(-0.60-2.20)(-0.60-1.28)(-0.60-0.66)(-0.60-0.00)(-0.60+1.04)(-0.60+1.20)}(1.04) \\
&+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.20)}{(-1.04-2.20)(-1.04-1.28)(-1.04-0.66)(-1.04-0.00)(-1.04+0.60)(-1.04+1.20)}(0.60) \\
&+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)}{(-1.20-2.20)(-1.20-1.28)(-1.20-0.66)(-1.20-0.00)(-1.20+0.60)(-1.20+1.04)}(0.00)
\end{aligned}$$

$$= \frac{x^6 - 0.02x^5 - 4.0784x^4 - 2.5406x^3 + 1.6220x^2 + 1.0873x}{-8.9744}$$
$$+ \frac{x^6 - 0.64x^5 - 4.4752x^4 - 0.27392x^3 + 4.6932x^2 + 2.1086x}{2.2023}$$
$$+ \frac{x^6 - 1.3x^5 - 4.0528x^4 + 2.6797x^3 + 4.8740x^2 - 0.98892x - 1.3917}{-1.1597}$$
$$+ \frac{x^6 - 1.9x^5 - 2.9128x^4 + 4.4274x^3 + 2.2176x^2 - 2.3195x}{1.0102}$$
$$+ \frac{x^6 - 2.34x^5 - 1.6192x^4 + 4.3637x^3 + 0.33581x^2 - 1.3382x}{-1.5593}$$

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 + 0.072013x^4 + 0.45241x^5 + 0.17103x^6, \quad -1.20 \le x \le 2.20$$

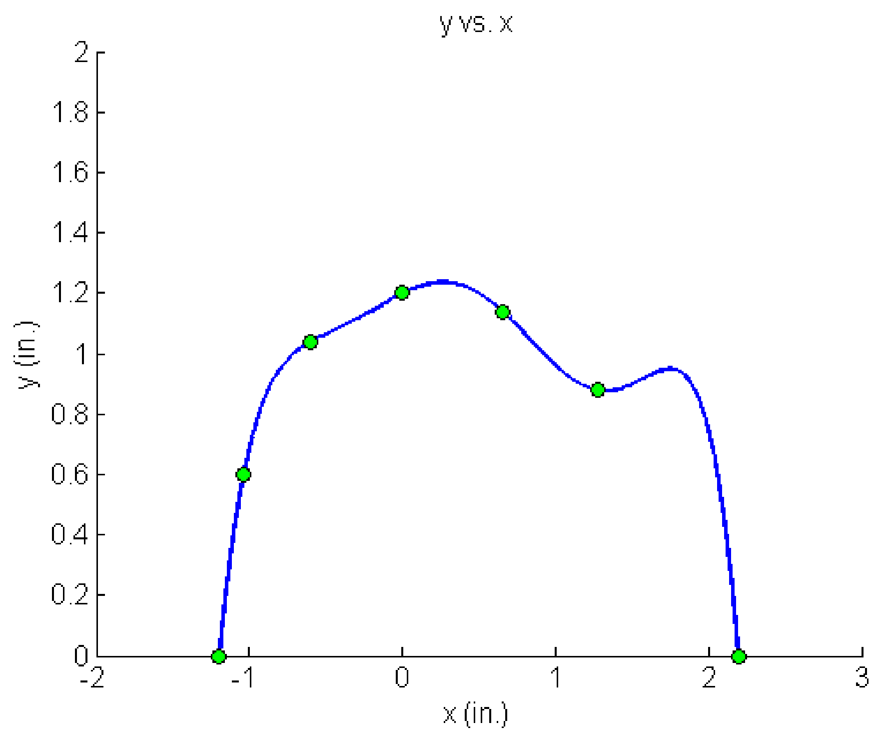

Figure 7 Plot of the cam profile as defined by a 6th order interpolating polynomial (using Lagrangian method of interpolation).