Chapter 05.02
Direct Method of Interpolation – More Examples
Industrial Engineering

Example 1
The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

Figure 1  Schematic of cam profile.

Table 1  Geometry of the cam.

<table>
<thead>
<tr>
<th>Point</th>
<th>( x ) (in.)</th>
<th>( y ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>-0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>-1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>-1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

If the cam follows a straight line profile from \( x = 1.28 \) to \( x = 0.66 \), what is the value of \( y \) at \( x = 1.10 \) using the direct method of interpolation and a first order polynomial?

Solution
For first order polynomial interpolation (also called linear interpolation), we choose the value of \( y \) given by
\[ y(x) = a_0 + a_1 x \]

\[ y \]

\[ f_i(x) \]

\[ (x_0, y_0) \]

\[ (x_i, y_i) \]

**Figure 2** Linear interpolation.

Since we want to find the value of \( y \) at \( x = 1.10 \), and we are using a first order polynomial, using the two points \( x_0 = 1.28 \) and \( x_i = 0.66 \), then

\[ x_0 = 1.28, \ y(x_0) = 0.88 \]
\[ x_i = 0.66, \ y(x_i) = 1.14 \]

gives

\[ y(1.28) = a_0 + a_i(1.28) = 0.88 \]
\[ y(0.66) = a_0 + a_i(0.66) = 1.14 \]

Writing the equations in matrix form, we have

\[
\begin{bmatrix}
1 & 1.28 \\
1 & 0.66
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_i
\end{bmatrix} =
\begin{bmatrix}
0.88 \\
1.14
\end{bmatrix}
\]

Solving the above two equations gives,

\[ a_0 = 1.4168 \]
\[ a_i = -0.41935 \]

Hence

\[ y(x) = a_0 + a_i x \]
\[ = 1.4168 - 0.41935x, \ 0.66 \leq x \leq 1.28 \]
\[ y(1.10) = 1.4168 - 0.41935(1.10) \]
\[ = 0.95548 \text{ in.} \]
Example 2

The geometry of a cam is given in Figure 3. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

![Figure 3 Schematic of cam profile.](image)

Table 2  Geometry of the cam.

<table>
<thead>
<tr>
<th>Point</th>
<th>x (in.)</th>
<th>y (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>−0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>−1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>−1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

If the cam follows a quadratic profile from $x = 2.20$ to $x = 1.28$ to $x = 0.66$, what is the value of $y$ at $x = 1.10$ using the direct method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the value of $y$ given by

$$y(x) = a_0 + a_1x + a_2x^2$$
Since we want to find the value of $y$ at $x = 1.10$, and we are using a second order polynomial, using the three points $x_0 = 2.20$, $x_1 = 1.28$ and $x_2 = 0.66$, then $x_0 = 2.20, \ y(x_0) = 0.00$

$x_1 = 1.28, \ y(x_1) = 0.88$

$x_2 = 0.66, \ y(x_2) = 1.14$

gives

$y(2.20) = a_0 + a_1(2.20) + a_2(2.20)^2 = 0.00$

$y(1.28) = a_0 + a_1(1.28) + a_2(1.28)^2 = 0.88$

$y(0.66) = a_0 + a_1(0.66) + a_2(0.66)^2 = 1.14$

Writing the three equations in matrix form, we have

$$
\begin{pmatrix}
1 & 2.20 & 4.84 \\
1 & 1.28 & 1.6384 \\
1 & 0.66 & 0.4356
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2
\end{pmatrix}
= 
\begin{pmatrix}
0.00 \\
0.88 \\
1.14
\end{pmatrix}
$$

Solving the above three equations gives

$a_0 = 1.1221$

$a_1 = 0.25734$

$a_2 = -0.34881$

Hence

$y(x) = 1.1221 + 0.25734x - 0.34881x^2, \ 0.66 \leq x \leq 2.20$

At $x = 1.10$,

$y(1.10) = 1.1221 + 0.25734(1.10) - 0.34881(1.10)^2$

$\quad = 0.98311$ in
The absolute relative approximate error \( |\varepsilon_a| \) obtained between the results from the first and second order polynomial is

\[
|\varepsilon_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100 = 2.8100\%
\]

**Example 3**

The geometry of a cam is given in Figure 5. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.

![Figure 5 Schematic of cam profile.](image)

**Table 3** Geometry of the cam.

<table>
<thead>
<tr>
<th>Point</th>
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<th>( y ) (in.)</th>
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<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
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<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>-0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>-1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>-1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Find the cam profile using all seven points in Table 3 using the direct method of interpolation and a sixth order polynomial.

**Solution**

For the sixth order polynomial, we choose the value of \( y \) given by

\[
y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6
\]
Figure 6 6th order polynomial interpolation.

Using the seven points,
\[ x_0 = 2.20, \quad y(x_0) = 0 \]
\[ x_1 = 1.28, \quad y(x_1) = 0.88 \]
\[ x_2 = 0.66, \quad y(x_2) = 1.14 \]
\[ x_3 = 0.00, \quad y(x_3) = 1.20 \]
\[ x_4 = -0.60, \quad y(x_4) = 1.04 \]
\[ x_5 = -1.04, \quad y(x_5) = 0.60 \]
\[ x_6 = -1.20, \quad y(x_6) = 0 \]

gives
\[
y(2.20) = 0.00 = a_0 + a_1(2.20) + a_2(2.20)^2 + a_3(2.20)^3 + a_4(2.20)^4 + a_5(2.20)^5 + a_6(2.20)^6
\]
\[
y(1.28) = 0.88 = a_0 + a_1(1.28) + a_2(1.28)^2 + a_3(1.28)^3 + a_4(1.28)^4 + a_5(1.28)^5 + a_6(1.28)^6
\]
\[
y(0.66) = 1.14 = a_0 + a_1(0.66) + a_2(0.66)^2 + a_3(0.66)^3 + a_4(0.66)^4 + a_5(0.66)^5 + a_6(0.66)^6
\]
\[
y(0.00) = 1.20 = a_0 + a_1(0.00) + a_2(0.00)^2 + a_3(0.00)^3 + a_4(0.00)^4 + a_5(0.00)^5 + a_6(0.00)^6
\]
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\[
y(-0.60) = 1.04 = a_0 + a_1(-0.60) + a_2(-0.60)^2 + a_3(-0.60)^3 + a_4(-0.60)^4 \\
+ a_5(-0.60)^5 + a_6(-0.60)^6
\]

\[
y(-1.04) = 0.60 = a_0 + a_1(-1.04) + a_2(-1.04)^2 + a_3(-1.04)^3 + a_4(-1.04)^4 \\
+ a_5(-1.04)^5 + a_6(-1.04)^6
\]

\[
y(-1.20) = 0.00 = a_0 + a_1(-1.20) + a_2(-1.20)^2 + a_3(-1.20)^3 + a_4(-1.20)^4 \\
+ a_5(-1.20)^5 + a_6(-1.20)^6
\]

Writing the seven equations in matrix form, we have

\[
\begin{bmatrix}
1 & 2.20 & 2.20^2 & 2.20^3 & 2.20^4 & 2.20^5 & 2.20^6 \\
1 & 1.28 & 1.28^2 & 1.28^3 & 1.28^4 & 1.28^5 & 1.28^6 \\
1 & 0.66 & 0.66^2 & 0.66^3 & 0.66^4 & 0.66^5 & 0.66^6 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.60 & 0.60^2 & -0.60^3 & 0.60^4 & -0.60^5 & 0.60^6 \\
1 & -1.04 & 1.04^2 & -1.04^3 & 1.04^4 & -1.04^5 & 1.04^6 \\
1 & -1.20 & 1.20^2 & -1.20^3 & 1.20^4 & -1.20^5 & 1.20^6
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6
\end{bmatrix}
= 
\begin{bmatrix}
0.00 \\
0.88 \\
1.14 \\
1.20 \\
1.04 \\
0.60 \\
0.00
\end{bmatrix}
\]

Solving the above seven equations gives

\[a_0 = 1.2\]
\[a_1 = 0.25112\]
\[a_2 = -0.27255\]
\[a_3 = -0.56765\]
\[a_4 = 0.072013\]
\[a_5 = 0.45241\]
\[a_6 = -0.17103\]

Hence

\[
y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6
\]

\[
= 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \\
+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \leq x \leq 2.20
\]
Figure 7  Plot of the cam profile as defined by a 6th order interpolating polynomial (using directed method of interpolation).