Chapter 04.08
Gauss-Seidel Method – More Examples
Industrial Engineering

Example 1

To find the number of toys a company should manufacture per day to optimally use their injection-molding machine and the assembly line, one needs to solve the following set of equations. The unknowns are the number of toys for boys, x_1, the number of toys for girls, x_2, and the number of unisexual toys, x_3.

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0.1667 & 0.6667 & 0.3333 \\ 1.05 & -1.00 & 0.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 1260 \\ 0 \end{bmatrix}$$

Find the values of x_1, x_2, and x_3 using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$x_1 = \frac{756 - 0.1667x_2 - 0.6667x_3}{0.3333}$$

$$x_2 = \frac{1260 - 0.1667x_1 - 0.3333x_3}{0.6667}$$

$$x_3 = \frac{0 - 1.05x_1 - (-1.00)x_2}{0}$$

The equation for x_3 is divided by 0 which is undefined. Therefore the order of the equations will need to be changed. Equation 3 and Equation 1 will be switched. By switching Equations 3 and 1, the matrix will also become diagonally dominant.

The system of equations becomes

$$\begin{bmatrix} 1.05 & -1.00 & 0.00 \\ 0.1667 & 0.6667 & 0.3333 \\ 0.3333 & 0.1667 & 0.6667 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1260 \\ 756 \end{bmatrix}$$

Rewriting the equations gives

$$x_1 = \frac{0-(-1.00)x_2 - 0x_3}{1.05}$$
$$x_2 = \frac{1260-0.1667x_1-0.3333x_3}{0.6667}$$
$$x_3 = \frac{756-0.3333x_1-0.1667x_2}{0.6667}$$

<u>Iteration #1</u>

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 100 \end{bmatrix}$$

we get

$$x_1 = \frac{0-(-1.00)\times 1000 - 0\times 100}{1.05}$$
$$= 952.38$$
$$x_2 = \frac{1260-0.1667\times 952.38-0.3333\times 100}{0.6667}$$
$$= 1601.8$$
$$x_3 = \frac{756-0.3333\times 952.38-0.1667\times 1601.8}{0.6667}$$
$$= 257.32$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{952.38-1000}{952.38}\right|\times 100$$
$$= 5\%$$

$$|\in_a|_2 = \left|\frac{1601.8-1000}{1601.8}\right|\times 100$$

$$= 37.570\%$$

$$|\in_a|_3 = \left|\frac{257.32-100}{257.32}\right|\times 100$$
$$= 61.138$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 952.38 \\ 1601.8 \\ 257.32 \end{bmatrix}$$

and the maximum absolute relative approximate error is 61.138%.

<u>Iteration #2</u>

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 952.38 \\ 1601.8 \\ 257.32 \end{bmatrix}$$

Now we get

$$x_1 = \frac{0-(-1.00)\times 1601.8 - 0\times 257.32}{1.05}$$
$$= 1525.5$$
$$x_2 = \frac{1260-0.1667\times 1525.5-0.3333\times 257.32}{0.6667}$$
$$= 1379.8$$
$$x_3 = \frac{756-0.3333\times 1525.5-0.1667\times 1379.8}{0.6667}$$
$$= 26.295$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{1525.5-952.38}{1525.5}\right|\times 100$$
$$= 37.570\%$$
$$\left|\in_a\right|_2 = \left|\frac{1379.8-1601.8}{1379.8}\right|\times 100$$
$$= 16.085\%$$
$$\left|\in_a\right|_3 = \left|\frac{26.295-257.32}{26.295}\right|\times 100$$
$$= 878.59\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1525.5 \\ 1379.8 \\ 26.295 \end{bmatrix}$$

and the maximum absolute relative approximate error is 878.59%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	952.38	5	1601.8	37.570	257.32	61.138
2	1525.5	37.570	1379.8	16.085	26.295	878.59
3	1314.1	16.085	1548.2	10.874	89.876	70.743
4	1474.5	10.874	1476.3	4.8686	27.694	224.53
5	1406.0	4.8686	1524.5	3.1618	49.863	44.459
6	1451.9	3.1618	1501.9	1.5021	32.554	53.170

After six iterations, the absolute relative approximate errors are decreasing, but they are still high. Allowing for more iterations, the absolute relative approximate errors decrease significantly.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
20	1439.8	0.00064276	1511.8	0.00034987	36.115	0.0091495
21	1439.8	0.00034987	1511.8	0.00019257	36.114	0.0049578

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1439.8 \\ 1511.8 \\ 36.113 \end{bmatrix}$$

SIMULTANEOUS LINEAR EQUATIONS	
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Authors	Autar Kaw
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