Chapter 04.07
LU Decomposition – More Examples
Industrial Engineering

Example 1

To find the number of toys a company should manufacture per day to optimally use their injection-molding machine and the assembly line, one needs to solve the following set of equations. The unknowns are the number of toys for boys, x_1, the number of toys for girls, x_2, and the number of unisexual toys, x_3.

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0.1667 & 0.6667 & 0.3333 \\ 1.05 & -1.00 & 0.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 1260 \\ 0 \end{bmatrix}$$

Find the values of x_1, x_2, and x_3 using naïve Gauss elimination.

Solution

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The $[U]$ matrix is the same as the one found at the end of the forward elimination steps of the naïve Gauss elimination method.

Forward Elimination of Unknowns

Since there are three equations, there will be two steps of forward elimination of unknowns.

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0.1667 & 0.6667 & 0.3333 \\ 1.05 & -1.00 & 0.00 \end{bmatrix}$$

<u>First step</u>

Divide Row 1 by 0.3333 and multiply it by 0.1667, that is, multiply Row 1 by $0.1667/0.3333 = 0.50015$. Then subtract the results from Row 2.

$$\text{Row } 1 \times (0.50015) = \begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.58332 & -0.00015002 \\ 1.05 & -1.00 & 0.00 \end{bmatrix}$$

Divide Row 1 by 0.3333 and multiply it by 1.05, that is, multiply Row 1 by $1.05/0.3333 = 3.1503$. Then subtract the results from Row 3.

$$\text{Row } 1 \times (3.1503) = \begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.58332 & -0.00015002 \\ 0 & -1.5252 & -2.1003 \end{bmatrix}$$

Second step

We now divide Row 2 by 0.58332 and multiply it by -1.5252, that is, multiply Row 2 by $-1.5252/0.58332 = -2.6146$. Then subtract the results from Row 3.

$$\text{Row } 2 \times (-2.6146) = \begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.58332 & -0.00015002 \\ 0 & 0 & -2.1007 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.58332 & -0.00015002 \\ 0 & 0 & -2.1007 \end{bmatrix}$$

Now find $[L]$.

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

From the first step of forward elimination,

$$\ell_{21} = \frac{0.1667}{0.3333} = 0.50015$$

$$\ell_{31} = \frac{1.05}{0.3333} = 3.1503$$

From the second step of forward elimination,

$$\ell_{32} = \frac{-1.5252}{0.58332} = -2.6146$$

Hence

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.50015 & 1 & 06 \\ 3.1503 & -2.6146 & 1 \end{bmatrix}$$

Now that $[L]$ and $[U]$ are known, solve $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.50015 & 1 & 0 \\ 3.1503 & -2.6146 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 1260 \\ 0 \end{bmatrix}$$

to give

$$z_1 = 756$$

$$0.50015 z_1 + z_2 = 1260$$

$$3.1503 z_1 + (-2.6146) z_2 + z_3 = 0$$

Forward substitution starting from the first equation gives

$$z_1 = 756$$

$$z_2 = 1260 - 0.50015z_1$$
$$= 1260 - 0.50015 \times 756$$
$$= 881.89$$
$$z_3 = 0 - 3.1503z_1 - (-2.6147)z_2$$
$$= 0 - 3.1503 \times 756 - (-2.6147) \times 881.89$$
$$= -75.864$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 881.89 \\ -75.864 \end{bmatrix}$$

Now solve $[U][X] = [Z]$.

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.58332 & -0.00015002 \\ 0 & 0 & -2.1007 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 881.89 \\ -75.864 \end{bmatrix}$$

$$0.3333x_1 + 0.1667x_2 + 0.6667x_3 = 756$$
$$0.58332x_2 + (-0.00015002)x_3 = 881.89$$
$$-2.1007x_3 = -75.864$$

From the third equation,

$$-2.1007x_3 = -75.864$$
$$x_3 = \frac{-75.864}{-2.1007}$$
$$= 36.113$$

Substituting the value of x_3 in the second equation,

$$0.58332x_2 + (-0.00015002)x_3 = 881.89$$
$$x_2 = \frac{881.89 - (-0.00015002)x_3}{0.58332}$$
$$= \frac{881.89 - (-0.00015002) \times 36.113}{0.58332}$$
$$= 1511.8$$

Substituting the values of x_2 and x_3 in the first equation,

$$0.3333x_1 + 0.1667x_2 + 0.6667x_3 = 756$$
$$x_1 = \frac{756 - 0.1667x_2 - 0.6667x_3}{0.3333}$$
$$= \frac{756 - 0.1667 \times 1511.8 - 0.6667 \times 36.113}{0.3333}$$
$$= 1439.8$$

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1439.8 \\ 1511.9 \\ 36.113 \end{bmatrix}$$

SIMULTANEOUS LINEAR EQUATIONS	
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Summary	Examples of LU decomposition
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