Chapter 02.03
Differentiation of Discrete Functions-More Examples
Industrial Engineering

Example 1

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 1.

Table 1 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using the forward divided difference method, find the failure rate of the DMFC system at $t = 50$ hours.

Solution

$$R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t}$$

$$t_i = 10$$

$$t_{i+1} = 100$$

$$\begin{aligned}\Delta t &= t_{i+1} - t_i \\ &= 100 - 10 \\ &= 90\end{aligned}$$

$$\begin{aligned}R'(50) &\approx \frac{R(100) - R(10)}{90} \\ &= \frac{0.9980 - 0.9998}{90} \\ &= -2.0000 \times 10^{-5}\end{aligned}$$

The reliability $R(t)$ at $t = 50$ hours is,

$$R(50) \approx \frac{R(100) - R(10)}{100 - 10}(50 - 10) + R(10)$$
$$= (-2.0000 \times 10^{-5})(40) + 0.9998$$
$$= 0.999$$

The failure rate $h(t)$ at $t = 50$ hours is then,

$$h(50) = -\frac{R'(50)}{R(50)}$$
$$h(50) = -\frac{(-2.0000 \times 10^{-5})}{0.999}$$
$$h(50) = 2.0020 \times 10^{-5}$$

Example 2

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 2.

Table 2 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using a third order polynomial interpolant for reliability $R(t)$, find the failure rate of the DMFC at $t = 50$ hours.

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the reliability given by

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the reliability at $t = 50$, and we are using a third order polynomial, we need to choose the four points closest to $t = 50$ that also bracket $t = 50$ to evaluate it.

The four points are $t_0 = 1$, $t_1 = 10$, $t_2 = 100$, and $t_3 = 1000$ hours.

$$t_0 = 1, \quad R(t_0) = 0.9999$$

$$t_1 = 10, \quad R(t_1) = 0.9998$$

$t_2 = 100, \quad R(t_2) = 0.9980$

$t_3 = 1000, \quad R(t_3) = 0.9802$

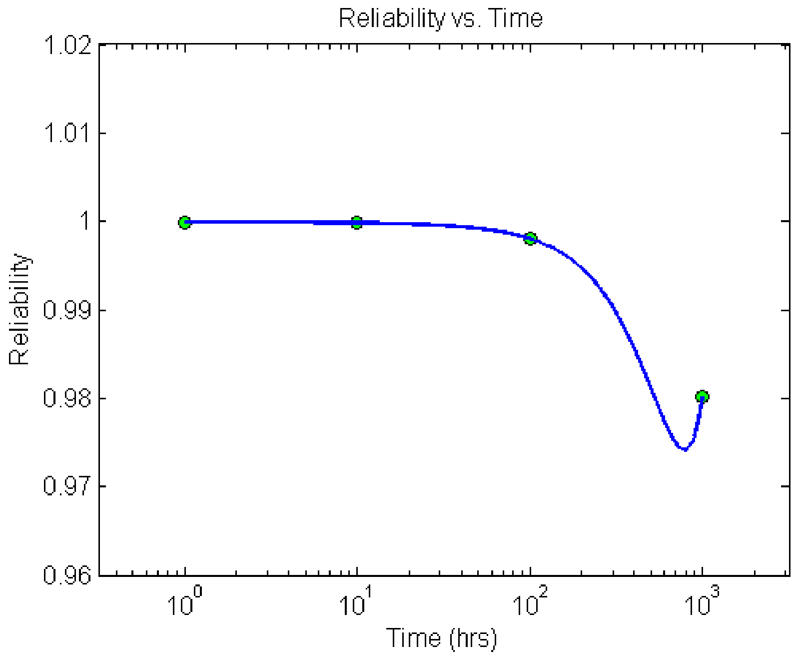

Figure 1 Graph of reliability as a function of time.

such that

$$R(1) = 0.9999 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$
$$R(10) = 0.9998 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$
$$R(100) = 0.9980 = a_0 + a_1(100) + a_2(100)^2 + a_3(100)^3$$
$$R(1000) = 0.9802 = a_0 + a_1(1000) + a_2(1000)^2 + a_3(1000)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 10000 & 1\times10^6 \\ 1 & 1000 & 1\times10^6 & 1\times10^9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.9999 \\ 0.9998 \\ 0.9980 \\ 0.9802 \end{bmatrix}$$

Solving the above gives

$a_0 = 0.99991$

$$a_1 = -1.0023\times 10^{-5}$$

$$a_2 = -9.9788\times 10^{-8}$$

$$a_3 = 9.0101\times 10^{-11}$$

Hence

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$= 0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3, \quad 1 \le t \le 1000$$

The acceleration at $t = 50$ is given by

$$R'(50) = \frac{d}{dt} R(t)\Big|_{t=50}$$

Given that $R(t) = 0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3, \quad 1 \le t \le 1000$,

$$R'(t) = \frac{d}{dt} R(t)$$

$$= \frac{d}{dt}\left(0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3\right)$$

$$= -1.0023\times 10^{-5} - 1.9958\times 10^{-7} t + 2.7030\times 10^{-10} t^2, \quad 1 \le t \le 1000$$

$$R'(50) = -1.0023\times 10^{-5} - 1.9958\times 10^{-7}(50) + 2.7030\times 10^{-10}(50)^2$$

$$= -1.9326\times 10^{-5}$$

Using the same function, we can also calculate the value of $R(t)$ at $t = 50$.

$$R(t) = 0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3, \quad 1 \le t \le 1000$$

$$R(50) = 0.99991 - 1.0023\times 10^{-5}(50) - 9.9788\times 10^{-8}(50)^2 + 9.0101\times 10^{-11}(50)^3$$

$$= 0.99917$$

The failure rate is then

$$h(t) = -\frac{R'(t)}{R(t)}$$

$$= -\frac{(-1.9326\times 10^{-5})}{0.99917}$$

$$= 1.9343\times 10^{-5}$$

Example 3

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 3.

Table 3 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Determine the value of the failure rate at $t = 50$ hours using second order Lagrangian polynomial interpolation for reliability.

Solution

For second order Lagrangian polynomial interpolation, we choose the reliability given by

$$R(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)R(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)R(t_1) + \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)R(t_2)$$

Since we want to find the reliability at $t = 50$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t = 50$ that also bracket $t = 50$ to evaluate it. The three points are $t_0 = 1$, $t_1 = 10$, and $t_2 = 100$.

Differentiating the above equation gives

$$R'(t) = \frac{2t - (t_1 + t_2)}{(t_0 - t_1)(t_0 - t_2)}R(t_0) + \frac{2t - (t_0 + t_2)}{(t_1 - t_0)(t_1 - t_2)}R(t_1) + \frac{2t - (t_0 + t_1)}{(t_2 - t_0)(t_2 - t_1)}R(t_2)$$

Hence

$$R'(50) = \frac{2(50) - (10 + 100)}{(1 - 10)(1 - 100)}(0.9999) + \frac{2(50) - (1 + 100)}{(10 - 1)(10 - 100)}(0.9998)$$

$$+ \frac{2(50) - (1 + 10)}{(100 - 1)(100 - 10)}(0.9980)$$

$$= -1.9102 \times 10^{-5}$$

We must also find the value of $R(t)$ at $t = 50$.

$$R(50) = \left(\frac{50 - 10}{1 - 10}\right)\left(\frac{50 - 100}{1 - 100}\right)(0.9999) + \left(\frac{50 - 1}{10 - 1}\right)\left(\frac{50 - 100}{10 - 100}\right)(0.9998)$$

$$+ \left(\frac{50 - 1}{100 - 1}\right)\left(\frac{50 - 10}{100 - 10}\right)(0.9980)$$

$$= 0.99918$$

The failure rate is then

$$h(t) = -\frac{R'(t)}{R(t)}$$

$$h(50) = -\frac{\left(-1.9102\times10^{-5}\right)}{0.99918}$$
$$= 1.9118\times10^{-5}$$

DIFFERENTIATION	
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Authors	Autar Kaw
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