Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Computer Engineering

Example 1

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150\ \mu\text{F}$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{\left|18\cos(120\pi(t))\right| - 2 - v(t)}{0.04}, 0 \right) \right\}$$
$$v(0) = 0$$

Using the Runge-Kutta 4th order method, find voltage across the capacitor at $t = 0.00004\,\text{s}$. Use step size $h = 0.00002\,\text{s}$.

Solution

$$\frac{dv}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{\left|18\cos(120\pi(t))\right| - 2 - v}{0.04}, 0 \right) \right\}$$

$$f(t, v) = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{\left|18\cos(120\pi(t))\right| - 2 - v}{0.04}, 0 \right) \right\}$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $v_0 = 0$

$$\begin{aligned} k_1 &= f(t_0, v_0) \\ &= f(0, 0) \\ &= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{\left|18\cos(120\pi(0))\right| - 2 - (0)}{0.04}, 0 \right) \right\} \\ &= \frac{1}{150 \times 10^{-6}} \{-0.1 + \max(400, 0)\} \end{aligned}$$

$$= \frac{1}{150\times10^{-6}}\{-0.1+400\}$$
$$= 2.6660\times10^{6}$$
$$k_2 = f\left(t_0+\frac{1}{2}h, v_0+\frac{1}{2}k_1h\right)$$
$$= f\left(0+\frac{1}{2}(0.00002), 0+\frac{1}{2}\left(2.6660\times10^{6}\right)0.00002\right)$$
$$= f(0.00001, 26.660)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00001))|-2-(26.660)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-266.50, 0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1-266.50\}$$
$$= -666.67$$
$$k_3 = f\left(t_0+\frac{1}{2}h, v_0+\frac{1}{2}k_2h\right)$$
$$= f\left(0+\frac{1}{2}(0.00002), 0+\frac{1}{2}(-666.67)0.00002\right)$$
$$= f(0.00001, -0.0066667)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00001))|-2-(-0.0066667)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(400.16, 0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+400.16\}$$
$$= 2.6671\times10^{6}$$
$$k_4 = f(t_0+h, v_0+k_3h)$$
$$= f\left(0+0.00002, 0+\left(2.6671\times10^{6}\right)0.00002\right)$$
$$= f(0.00002, 53.342)$$
$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00002))|-2-(53.342)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-933.56, 0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 0 + \frac{1}{6}\left(2.6660\times 10^6 + 2(-666.67) + 2\left(2.6671\times 10^6\right) + (-666.67)\right)0.00002$$
$$= 0 + \frac{1}{6}\left(7.9982\times 10^6\right)0.00002$$
$$= 26.661 \text{ V}$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$
$$v(0.00002) \approx v_1 = 26.661\,\text{V}$$

For $i = 1, t_1 = 0.00002, v_1 = 26.661$

$$k_1 = f(t_1, v_1)$$
$$= f(0.00002, 26.661)$$
$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00002))| - 2 - (26.661)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times 10^{-6}}\{-0.1 + \max(-266.51, 0)\}$$
$$= \frac{1}{150\times 10^{-6}}\{-0.1 + 0\}$$
$$= -666.67$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$
$$= f(0.00003, 26.654)$$
$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00003))| - 2 - (26.654)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times 10^{-6}}\{-0.1 + \max(-266.35, 0)\}$$
$$= \frac{1}{150\times 10^{-6}}\{-0.1 + 0\}$$
$$= -666.67$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$
$$= f(0.00003, 26.654)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(0.00003))\right|-2-(26.654)}{0.04},0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-266.35,0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$k_4 = f(t_1+h, v_1+k_3h)$$
$$= f(0.00002+(0.00002), 26.661+(-666.67)0.00002)$$
$$= f(0.00003, 26.647)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(0.00003))\right|-2-(26.634)}{0.04},0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-265.87,0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$v_2 = v_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$
$$= 26.661+\frac{1}{6}(-666.67+2(-666.67)+2(-666.67)+(-666.67))0.00002$$
$$= 26.661+\frac{1}{6}(-4000.0)0.00002$$
$$= 26.647\text{ V}$$

v_2 is the approximate voltage at $t = t_2$

$$t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004\text{ s}$$
$$v(.00004) \approx v_2 = 26.647\text{ V}$$

Figure 1 compares the exact solution of $v(0.00004) = 15.974\text{ V}$ with the numerical solution using Runge-Kutta 4th order method step size of $h = 0.00002\text{ s}$.

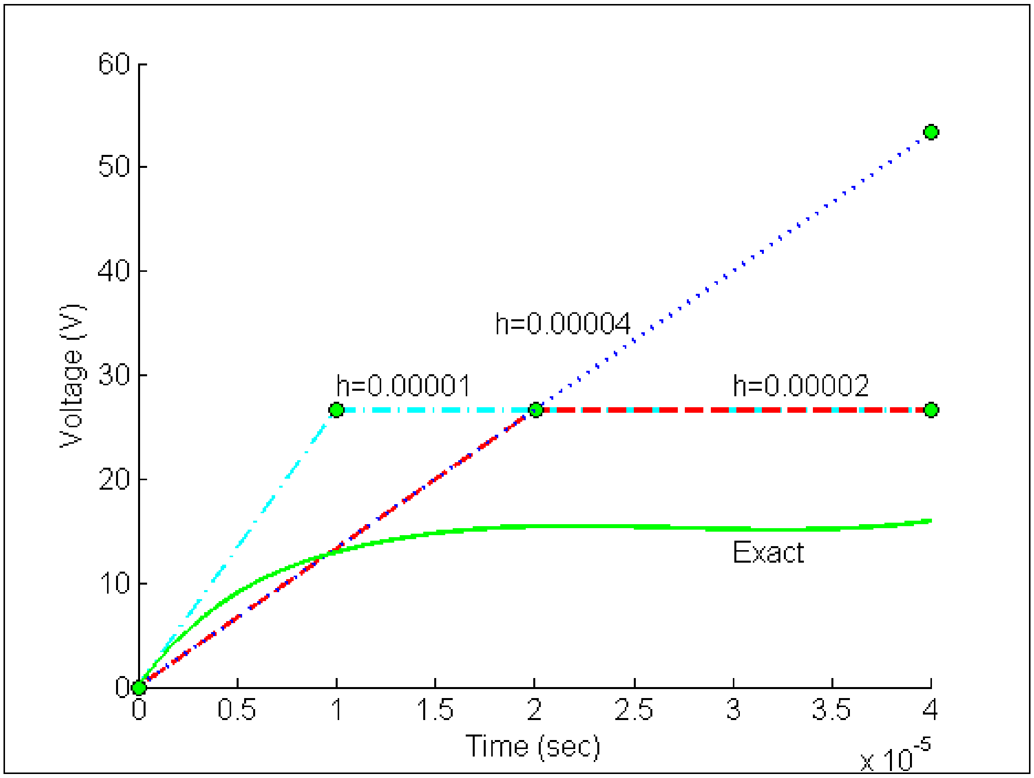

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at $t = 0.00004\,\text{s}$.

Table 1 Value of voltage at time, $t = 0.00004\,\text{s}$ for different step sizes.

Step size, h	$v(0.00004)$	E_t	$\lvert \in_t \rvert \%$
0.00004	53.335	−37.361	233.89
0.00002	26.647	−10.673	66.817
0.00001	15.986	−0.012299	0.076996
0.000005	15.975	−0.00050402	0.0031552
0.0000025	15.976	−0.0015916	0.0099639

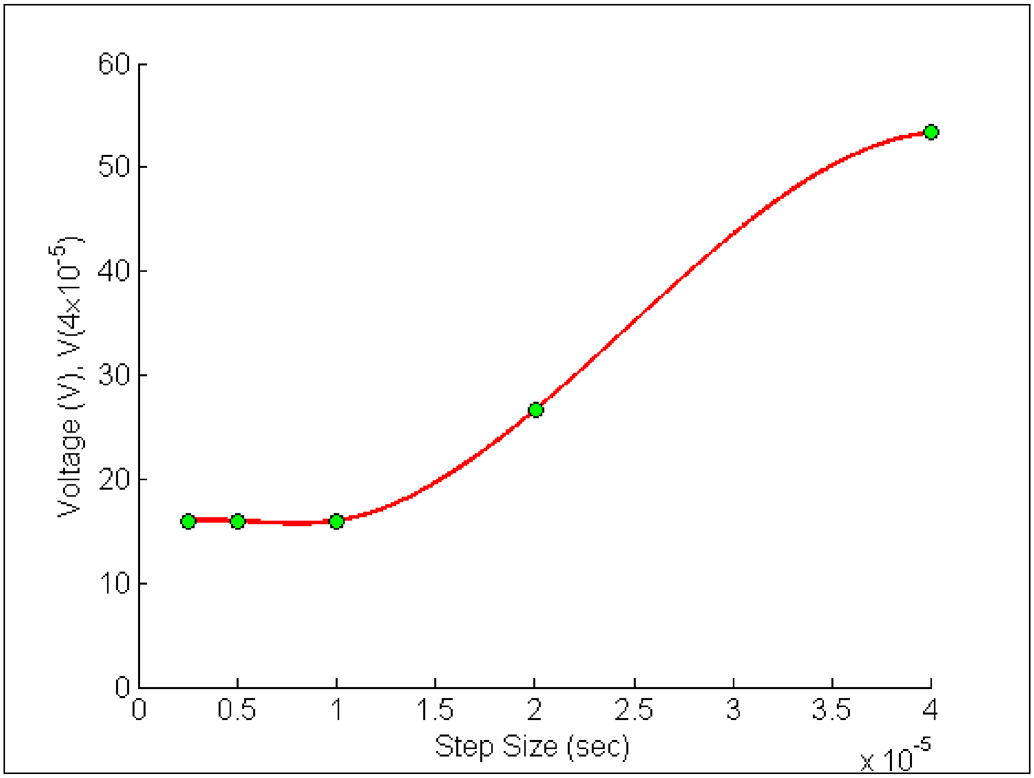

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.

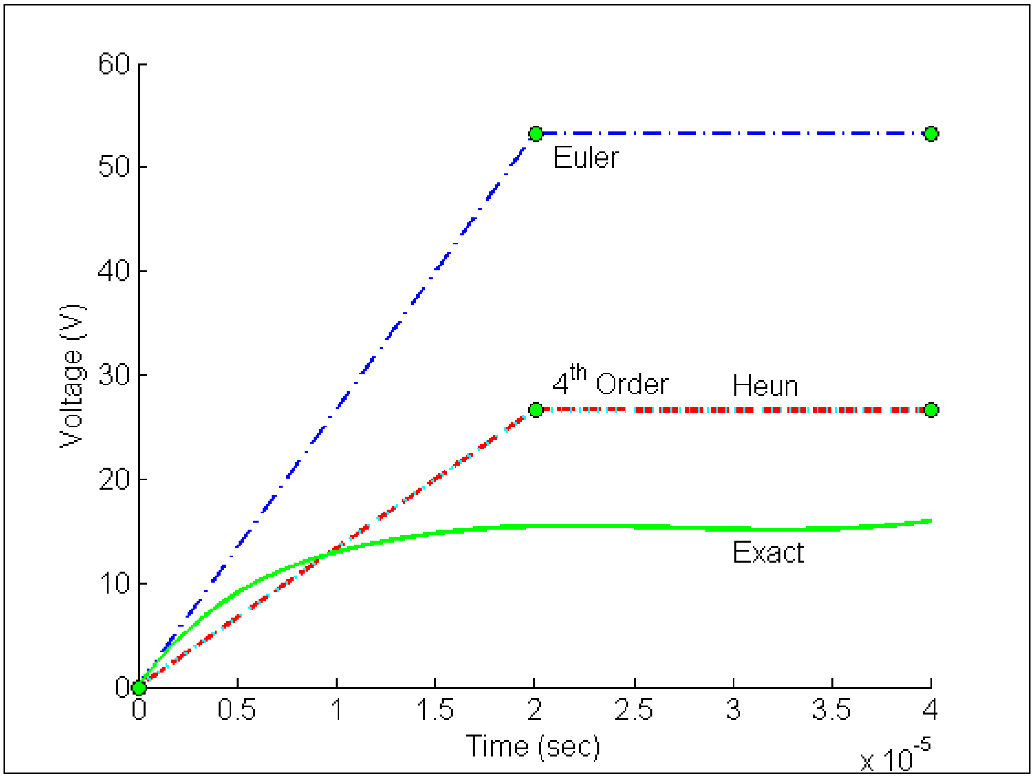

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.