Chapter 07.05
Romberg Rule for Integration-More Examples
Electrical Engineering

Example 1

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Table 1 Values obtained for Trapezoidal rule.

n	Trapezoidal Rule
1	0.11489
2	0.99637
4	0.96969
8	0.97901

a) Use Richardson's extrapolation formula to find the frequency. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $I_2 = 0.99637$

$I_4 = 0.96969$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$TV \approx I_4 + \frac{I_4 - I_2}{3}$$

$$\approx 0.96969 + \frac{0.96969 - 0.99637}{3}$$
$$\approx 0.96078$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value.

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= 0.98236$$

So the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 0.98236 - 0.96078$$
$$= 0.021560$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{0.98236 - 0.96078}{0.98236}\right| \times 100\,\%$$
$$= 2.1947\,\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Table 2 Values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

n	Trapezoidal Rule	$\|\in_t\|$ for Trapezoidal Rule %	Richardson's Extrapolation	$\|\in_t\|$ for Richardson's Extrapolation %
1	0.11489	88.3	--	--
2	0.99637	1.427	1.2902	31.337
4	0.96969	1.289	0.96078	2.1947
8	0.97901	0.3404	0.98212	0.024422

Example 2

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

Use Romberg's rule to find the frequency. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1}=0.11489$$

$$I_{1,2}=0.99637$$

$$I_{1,3}=0.96969$$

$$I_{1,4}=0.97901$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$\begin{aligned}I_{2,1}&=I_{1,2}+\frac{I_{1,2}-I_{1,1}}{3}\\&=0.99637+\frac{0.99637-0.11489}{3}\\&=1.2902\end{aligned}$$

Similarly

$$\begin{aligned}I_{2,2}&=I_{1,3}+\frac{I_{1,3}-I_{1,2}}{3}\\&=0.96969+\frac{0.96969-0.99637}{3}\\&=0.96080\end{aligned}$$

$$\begin{aligned}I_{2,3}&=I_{1,4}+\frac{I_{1,4}-I_{1,3}}{3}\\&=0.97901+\frac{0.97901-0.96969}{3}\\&=0.98212\end{aligned}$$

For the second order extrapolation values,

$$I_{3,1}=I_{2,2}+\frac{I_{2,2}-I_{2,1}}{15}$$

$$= 0.96080 + \frac{0.96080 - 1.2902}{15}$$
$$= 0.93884$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$
$$= 0.98212 + \frac{0.98212 - 0.96080}{15}$$
$$= 0.98354$$

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$
$$= 0.98354 + \frac{0.98354 - 0.93884}{63}$$
$$= 0.98425$$

Table 2 shows these increased correct values in a tree graph.

Table 3 Improved estimates of value of integral using Romberg integration.

		1st Order	2nd Order	3rd Order
1-segment	0.11489			
		1.2902		
			0.93884	
2-segment	0.99637			
				0.98425
		0.96080		
			0.98354	
4-segment	0.96969			
		0.98212		
8-segment	0.97901			