

Chapter 07.06

Gauss Quadrature Rule for Integration-More Examples

Electrical Engineering

Example 1:

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1 - \alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

- Use two-point Gauss quadrature rule to find the frequency.
- Find the absolute relative true error.

Solution

a) First, change the limits of integration from $[-2.15, 2.9]$ to $[-1, 1]$ using

$$a = -2.15$$

$$b = 2.9$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

gives

$$\begin{aligned} \int_{-2.15}^{2.9} f(x) dx &= \frac{2.9 - (-2.15)}{2} \int_{-1}^1 f\left(\frac{2.9 - (-2.15)}{2}x + \frac{2.9 + (-2.15)}{2}\right) dx \\ &= 2.525 \int_{-1}^1 f(2.525x + 0.375) dx \end{aligned}$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = 1.0000$$

$$x_1 = -0.57735$$

$$c_2 = 1.0000$$

$$x_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$\begin{aligned} 2.525 \int_{-1}^1 f(2.525x + 0.375) dx &\approx 2.525 [c_1 f(2.525x_1 + 0.375) + c_2 f(2.525x_2 + 0.375)] \\ &\approx 2.525 [f(2.525(-0.57735) + 0.375) + f(2.525(0.57735) + 0.375)] \\ &\approx 2.525 [f(-1.0828) + f(1.8328)] \\ &\approx 2.525 [(0.22198) + (0.074383)] \\ &\approx 0.74831 \end{aligned}$$

since

$$\begin{aligned} f(-1.0828) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1.0828)^2}{2}} \\ &= 0.22198 \\ f(1.8328) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(1.8328)^2}{2}} \\ &= 0.074383 \end{aligned}$$

b) The absolute relative true error, $|\epsilon_t|$, is (Exact value = 0.98236)

$$\begin{aligned} |\epsilon_t| &= \left| \frac{0.98236 - 0.74831}{0.98236} \right| \times 100 \% \\ &= 23.825 \% \end{aligned}$$

Example 2

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1 - \alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

- Use three-point Gauss quadrature rule to find the frequency.
- Find the absolute relative true error.

Solution:

a) First, change the limits of integration from $[-2.15, 2.9]$ to $[-1, 1]$ using

$$a = -2.15$$

$$b = 2.9$$

$$\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\begin{aligned} \int_{-2.15}^{2.9} f(x)dx &= \frac{2.9 - (-2.15)}{2} \int_{-1}^1 f\left(\frac{2.9 - (-2.15)}{2}x + \frac{2.9 + (-2.15)}{2}\right)dx \\ &= 2.5250 \int_{-1}^1 f(2.5250x + 0.37500)dx \end{aligned}$$

The weighting factors and function argument values are

$$c_1 = 0.55556$$

$$x_1 = -0.77460$$

$$c_2 = 0.88889$$

$$x_2 = 0.0000$$

$$c_3 = 0.55556$$

$$x_3 = 0.77460$$

and the formula is

$$\begin{aligned} 2.5250 \int_{-1}^1 f(2.5250x + 0.37500)dx &\approx 2.5250 \left[c_1 f(2.5250x_1 + 0.37500) + c_2 f(2.5250x_2 + 0.37500) \right. \\ &\quad \left. + c_3 f(2.5250x_3 + 0.37500) \right] \\ &\approx 2.525 \left[0.55556 f(2.5250(-0.77460) + 0.37500) + 0.88889 f(2.5250(0.0000) + 0.37500) \right. \\ &\quad \left. + 0.55556 f(2.5250(0.77460) + 0.37500) \right] \\ &\approx 2.525 \left[0.55556 f(-1.5809) + 0.88889 f(0.37500) + 0.55556 f(2.3309) \right] \\ &\approx 2.525 \left[0.55556(0.11435) + 0.88889(0.37185) + 0.55556(0.026374) \right] \\ &\approx 1.0320 \end{aligned}$$

since

$$\begin{aligned} f(-1.5809) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1.5809)^2}{2}} \\ &= 0.11435 \end{aligned}$$

$$\begin{aligned} f(0.37500) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(0.37500)^2}{2}} \\ &= 0.37186 \end{aligned}$$

$$\begin{aligned} f(2.3309) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(2.3309)^2}{2}} \\ &= 0.026374 \end{aligned}$$

b) The absolute relative true error, $|\epsilon_t|$, is (Exact value = 0.98236)

$$\begin{aligned} |\epsilon_t| &= \left| \frac{0.98236 - 1.0320}{0.98236} \right| \times 100 \% \\ &= 5.0547 \% \end{aligned}$$