04.08
Gauss-Seidel Method – More Examples
Electrical Engineering

Example 1

Three-phase loads are common in AC systems. When the system is balanced the analysis can be simplified to a single equivalent circuit model. However, when it is unbalanced the only practical solution involves the solution of simultaneous linear equations. In one model the following equations need to be solved.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

Find the values of I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and I_{ci} using the Gauss-Seidel method. Use

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$I_{ar} = \frac{120 - (-0.4516)I_{ai} - 0.0100I_{br} - (-0.0080)I_{bi} - 0.0100I_{cr} - (-0.0080)I_{ci}}{0.7460}$$

$$I_{ai} = \frac{0.000 - 0.4516I_{ar} - 0.0080I_{br} - 0.0100I_{bi} - 0.0080I_{cr} - 0.0100I_{ci}}{0.7460}$$

$$I_{br} = \frac{-60.00 - 0.0100I_{ar} - (-0.0080)I_{ai} - (-0.5205)I_{bi} - 0.0100I_{cr} - (-0.0080)I_{ci}}{0.7787}$$

$$I_{bi} = \frac{-103.9 - 0.0080I_{ar} - 0.0100I_{ai} - 0.5205I_{br} - 0.0080I_{cr} - 0.0100I_{ci}}{0.7787}$$

$$I_{cr} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - (-0.6040) I_{ci}}{0.8080}$$

$$I_{ci} = \frac{103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.6040 I_{cr}}{0.8080}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

Substituting the guess values into the first equation

$$I_{ar} = \frac{120 - (-0.4516) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7460}$$

$$= 172.86$$

Substituting the new value of I_{ar} and the remaining guess values into the second equation

$$I_{ai} = \frac{0.000 - 0.4516 I_{ar} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7460}$$

$$= -105.61$$

Substituting the new values of I_{ar}, I_{ai}, and the remaining guess values into the third equation

$$I_{br} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080) I_{ai} - (-0.5205) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7787}$$

$$= -67.039$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, and the remaining guess values into the fourth equation

$$I_{bi} = \frac{-103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.5205 I_{br} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7787}$$

$$= -89.499$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, I_{bi}, and the remaining guess values into the fifth equation

$$I_{cr} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - (-0.6040) I_{ci}}{0.8080}$$

$$= -62.548$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and the remaining guess value into the sixth equation

$$I_{ci} = \frac{103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.6040 I_{cr}}{0.8080}$$

$$= 176.71$$

The absolute relative approximate error for each I then is

$$\left|\in_a\right|_1 = \left|\frac{172.86 - 20}{172.86}\right| \times 100$$
$$= 88.430\%$$
$$\left|\in_a\right|_2 = \left|\frac{-105.61 - 20}{-105.61}\right| \times 100$$
$$= 118.94\%$$
$$\left|\in_a\right|_3 = \left|\frac{-67.039 - 20}{-67.039}\right| \times 100$$
$$= 129.83\%$$
$$\left|\in_a\right|_4 = \left|\frac{-89.499 - 20}{-89.499}\right| \times 100$$
$$= 122.35\%$$
$$\left|\in_a\right|_5 = \left|\frac{-62.548 - 20}{-62.548}\right| \times 100$$
$$= 131.98\%$$
$$\left|\in_a\right|_6 = \left|\frac{176.71 - 20}{176.71}\right| \times 100$$
$$= 88.682\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 172.86 \\ -105.61 \\ -67.039 \\ -89.499 \\ -62.548 \\ 176.71 \end{bmatrix}$$

and the maximum absolute relative approximate error is 131.98%.

Iteration #2

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 172.86 \\ -105.61 \\ -67.039 \\ -89.499 \\ -62.548 \\ 176.71 \end{bmatrix}$$

Substituting the values from Iteration #1 into the first equation

$$I_{ar} = \frac{120-(-0.4516)I_{ai} - 0.0100I_{br} - (-0.0080)I_{bi} - 0.0100I_{cr} - (-0.0080)I_{ci}}{0.7460}$$

$$= 99.600$$

Substituting the new value of I_{ar} and the remaining values from Iteration #1 into the second equation

$$I_{ai} = \frac{0.000 - 0.4516I_{ar} - 0.0080I_{br} - 0.0100I_{bi} - 0.0080I_{cr} - 0.0100I_{ci}}{0.7460}$$

$$= -60.073$$

Substituting the new values of I_{ar}, I_{ai}, and the remaining values from Iteration #1 into the third equation

$$I_{br} = \frac{-60.00 - 0.0100I_{ar} - (-0.0080)I_{ai} - (-0.5205)I_{bi} - 0.0100I_{cr} - (-0.0080)I_{ci}}{0.7787}$$

$$= -136.15$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, and the remaining values from Iteration #1 into the fourth equation

$$I_{bi} = \frac{-103.9 - 0.0080I_{ar} - 0.0100I_{ai} - 0.5205I_{br} - 0.0080I_{cr} - 0.0100I_{ci}}{0.7787}$$

$$= -44.299$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, I_{bi}, and the remaining values from Iteration #1 into the fifth equation

$$I_{cr} = \frac{-60.00 - 0.0100I_{ar} - (-0.0080)I_{ai} - 0.0100I_{br} - (-0.0080)I_{bi} - (-0.6040)I_{ci}}{0.8080}$$

$$= 57.259$$

Substituting the new values of I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and the remaining value from Iteration #1 into the sixth equation

$$I_{ci} = \frac{103.9 - 0.0080I_{ar} - 0.0100I_{ai} - 0.0080I_{br} - 0.0100I_{bi} - 0.6040I_{cr}}{0.8080}$$

$$= 87.441$$

The absolute relative approximate error for each I then is

$$|\in_a|_1 = \left|\frac{99.600 - 172.86}{99.600}\right| \times 100$$

$$= 73.552\%$$

$$|\in_a|_2 = \left|\frac{-60.073 - (-105.61)}{-60.073}\right| \times 100$$

$$= 75.796\%$$

$$|\in_a|_3 = \left|\frac{-136.35 - (-67.039)}{-136.35}\right| \times 100$$

$$= 50.762\%$$

$$\left|\in_a\right|_4 = \left|\frac{-44.299-(-89.499)}{-44.299}\right|\times 100$$
$$= 102.03\%$$
$$\left|\in_a\right|_5 = \left|\frac{57.259-(-62.548)}{57.259}\right|\times 100$$
$$= 209.24\%$$
$$\left|\in_a\right|_6 = \left|\frac{87.441-176.71}{87.441}\right|\times 100$$
$$= 102.09\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 99.600 \\ -60.073 \\ -136.15 \\ -44.299 \\ 57.259 \\ 87.441 \end{bmatrix}$$

and the maximum absolute relative approximate error is 141.4087%.
Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	I_{ar}	I_{ai}	I_{br}	I_{bi}	I_{cr}	I_{ci}
1	172.86	–105.61	–67.039	–89.499	–62.548	176.71
2	99.600	–60.073	–136.15	–44.299	57.259	87.441
3	126.01	–76.015	–108.90	–62.667	–10.478	137.97
4	117.25	–70.707	–119.62	–55.432	27.658	109.45
5	119.87	–72.301	–115.62	–58.141	6.2513	125.49
6	119.28	–71.936	–116.98	–57.216	18.241	116.53

Iteration	$\left\|\in_a\right\|_1\%$	$\left\|\in_a\right\|_2\%$	$\left\|\in_a\right\|_3\%$	$\left\|\in_a\right\|_4\%$	$\left\|\in_a\right\|_5\%$	$\left\|\in_a\right\|_6\%$
1	88.430	118.94	129.83	122.35	131.98	88.682
2	73.552	75.796	50.762	102.03	209.24	102.09
3	20.960	20.972	25.027	29.311	646.45	36.623
4	7.4738	7.5067	8.9631	13.053	137.89	26.001
5	2.1840	2.2048	3.4633	4.6595	342.43	12.742
6	0.49408	0.50789	1.1629	1.6170	65.729	7.6884

After six iterations, the absolute relative approximate errors are decreasing, but are still high. Allowing for more iteration, the relative approximate errors decrease significantly.

Iteration	I_{ar}	I_{ai}	I_{br}	I_{bi}	I_{cr}	I_{ci}
32	119.33	–71.973	–116.66	–57.432	13.940	119.74
33	119.33	–71.973	–116.66	–57.432	13.940	119.74

Iteration	$\left\|\in_a\right\|_1 \%$	$\left\|\in_a\right\|_2 \%$	$\left\|\in_a\right\|_3 \%$	$\left\|\in_a\right\|_4 \%$	$\left\|\in_a\right\|_5 \%$	$\left\|\in_a\right\|_6 \%$
32	3.0666×10^{-7}	3.0047×10^{-7}	4.2389×10^{-7}	5.7116×10^{-7}	2.0941×10^{-5}	1.8238×10^{-6}
33	1.7062×10^{-7}	1.6718×10^{-7}	2.3601×10^{-7}	3.1801×10^{-7}	1.1647×10^{-5}	1.0144×10^{-6}

After 33 iterations, the solution vector is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 119.33 \\ -71.973 \\ -116.66 \\ -57.432 \\ 13.940 \\ 119.74 \end{bmatrix}$$

The maximum absolute relative approximate error is $1.1647\times10^{-5}\%$.

SIMULTANEOUS LINEAR EQUATIONS	
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