Chapter 03.03
Bisection Method of Solving a Nonlinear Equation – More Examples
Electrical Engineering

Example 1
Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature. For a 10K3A Betatherm thermistor,

![Thermistor Diagram](image)

**Figure 1** A typical thermistor.

the relationship between the resistance $R$ of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} |\ln(R)|^3$$

where $T$ is in Kelvin and $R$ is in ohms.

A thermistor error of no more than ± 0.01°C is acceptable. To find the range of the resistance that is within this acceptable limit at 19°C, we need to solve

$$\frac{1}{19.01 + 273.15} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} |\ln(R)|^3$$

and

$$\frac{1}{18.99 + 273.15} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} |\ln(R)|^3$$
Use the bisection method of finding roots of equations to find the resistance $R$ at 18.99°C. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

**Solution**

Solving

$$\frac{1}{18.99 + 273.15} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} \{\ln(R)\}^3$$

we get

$$f(R) = 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} \{\ln(R)\}^3 - 2.293775 \times 10^{-3}$$

Lets us assume $R_1 = 11000$, $R_u = 14000$.

Check if the function changes sign between $R_1$ and $R_u$.

$$f(R_1) = f(11000)$$

$$= 2.341077 \times 10^{-4} \ln(11000) + 8.775468 \times 10^{-8} \{\ln(11000)\}^3 - 2.293775 \times 10^{-3}$$

$$= -4.4536 \times 10^{-5}$$

$$f(R_u) = f(14000)$$

$$= 2.341077 \times 10^{-4} \ln(14000) + 8.775468 \times 10^{-8} \{\ln(14000)\}^3 - 2.293775 \times 10^{-3}$$

$$= 1.7563 \times 10^{-5}$$

Hence

$$f(R_1)f(R_u) = f(11000)f(14000) = \left( -4.4536 \times 10^{-5} \right) \left( 1.7563 \times 10^{-5} \right) < 0$$

So there is at least one root between $R_1$ and $R_u$, that is, between 11000 and 14000.

**Iteration 1**

The estimate of the root is

$$R_m = \frac{R_1 + R_u}{2}$$

$$= \frac{11000 + 14000}{2}$$

$$= 12500$$

$$f(R_m) = f(12500)$$

$$= 2.341077 \times 10^{-4} \ln(12500) + 8.775468 \times 10^{-8} \{\ln(12500)\}^3 - 2.293775 \times 10^{-3}$$

$$= -1.1655 \times 10^{-5}$$

$$f(R_1)f(R_m) = f(11000)f(12500) = \left( -4.4536 \times 10^{-5} \right) \left( -1.1655 \times 10^{-5} \right) > 0$$

Hence the root is bracketed between $R_m$ and $R_u$, that is, between 12500 and 14000. So, the lower and upper limits of the new bracket are $R_1 = 12500$, $R_u = 14000$. 


At this point, the absolute relative approximate error $|\varepsilon_a|$ cannot be calculated as we do not have a previous approximation.

**Iteration 2**
The estimate of the root is

$$R_m = \frac{R_l + R_u}{2} = \frac{12500 + 14000}{2} = 13250$$

$$f(R_m) = f(13250) = 2.341077 \times 10^{-4} \ln(13250) + 8.775468 \times 10^{-8} \ln(13250) - 2.293775 \times 10^{-3}$$

$$f(R_l) f(R_m) = f(12500) f(13250) = (-1.1655 \times 10^{-3}) (3.3599 \times 10^{-6}) < 0$$

Hence, the root is bracketed between $R_l$ and $R_m$, that is, between 12500 and 13250. So the lower and upper limits of the new bracket are $R_l = 12500$, $R_u = 13250$

The absolute relative approximate error $|\varepsilon_a|$ at the end of Iteration 2 is

$$|\varepsilon_a| = \left|\frac{R_m^{\text{new}} - R_m^{\text{old}}}{R_m^{\text{new}}}\right| \times 100 = \left|\frac{13250 - 12500}{13250}\right| \times 100 = 5.6604\%$$

None of the significant digits are at least correct in the estimated root $R_m = 13250$ as the absolute relative approximate error is greater than 5%.

**Iteration 3**

$$R_m = \frac{R_l + R_u}{2} = \frac{12500 + 13250}{2} = 12875$$

$$f(R_m) = f(12875) = 2.341077 \times 10^{-4} \ln(12875) + 8.775468 \times 10^{-8} \ln(12875) - 2.293775 \times 10^{-3}$$

$$f(R_l) f(R_m) = f(12500) f(12875) = (-1.1654 \times 10^{-5}) (-4.0398 \times 10^{-6}) > 0$$
Hence, the root is bracketed between $R_u$ and $R_u$, that is, between 12875 and 13250.

So, the lower and upper limits of the new bracket are

\[ R_u = 12875, \quad R_u = 13250 \]

The absolute relative approximate error $|\varepsilon_a|$ at the end of Iteration 3 is

\[
|\varepsilon_a| = \left| \frac{R_{m}^{\text{new}} - R_{m}^{\text{old}}}{R_{m}^{\text{new}}} \right| \times 100 = \frac{12875 - 13250}{12875} \times 100 = 2.9126\% 
\]

One of the significant digits is at least correct in the estimated root of the equation as the absolute relative approximate error is less than 5%.

Seven more iterations were conducted and these iterations are shown in the Table 1.

**Table 1** Root of $f(x) = 0$ as a function of the number of iterations for bisection method.

| Iteration | $R_u$   | $R_u$   | $R_m$   | $|\varepsilon_a|\%$ | $f(R_m)$   |
|-----------|---------|---------|---------|---------------------|------------|
| 1         | 11000   | 14000   | 12500   | 1.1655 $\times 10^{-5}$ |
| 2         | 12500   | 14000   | 13250   | 3.3599 $\times 10^{-6}$ |
| 3         | 12500   | 13250   | 12875   | -4.0403 $\times 10^{-6}$ |
| 4         | 12875   | 13250   | 13063   | -3.1417 $\times 10^{-7}$ |
| 5         | 13063   | 13250   | 13156   | 1.5293 $\times 10^{-6}$ |
| 6         | 13063   | 13156   | 13109   | 6.0917 $\times 10^{-7}$ |
| 7         | 13063   | 13109   | 13086   | 1.4791 $\times 10^{-7}$ |
| 8         | 13063   | 13086   | 13074   | -8.3022 $\times 10^{-8}$ |
| 9         | 13074   | 13086   | 13080   | 3.2470 $\times 10^{-8}$ |
| 10        | 13074   | 13080   | 13077   | -2.5270 $\times 10^{-8}$ |

At the end of the 10th iteration,

\[ |\varepsilon_a| = 0.022403\% \]

Hence the number of significant digits at least correct is given by the largest value of $m$ for which

\[ |\varepsilon_a| \leq 0.5 \times 10^{-m} \]

0.022403 $\leq 0.5 \times 10^{-2-m}$

0.044806 $\leq 10^{-2-m}$

$\log(0.044806) \leq 2 - m$

So

\[ m = 2 - \log(0.044806) = 3.3487 \]

So

\[ m = 3 \]

The number of significant digits at least correct in the estimated root 13077 is 3.
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