Chapter 03.03
Bisection Method of Solving a Nonlinear Equation – More Examples
Electrical Engineering

Example 1

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature. For a 10K3A Betatherm thermistor,

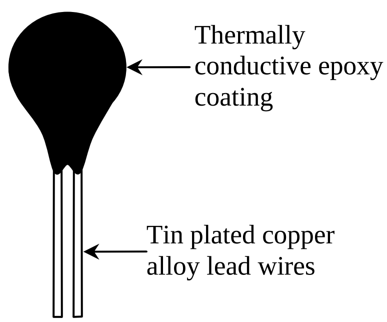

Figure 1 A typical thermistor.

the relationship between the resistance R of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241\times 10^{-3} + 2.341077\times 10^{-4}\ln(R) + 8.775468\times 10^{-8}\{\ln(R)\}^3$$

where T is in Kelvin and R is in ohms.

A thermistor error of no more than $\pm 0.01°\mathrm{C}$ is acceptable. To find the range of the resistance that is within this acceptable limit at $19\ °\mathrm{C}$, we need to solve

$$\frac{1}{19.01+273.15} = 1.129241\times 10^{-3} + 2.341077\times 10^{-4}\ln(R) + 8.775468\times 10^{-8}\{\ln(R)\}^3$$

and

$$\frac{1}{18.99+273.15} = 1.129241\times 10^{-3} + 2.341077\times 10^{-4}\ln(R) + 8.775468\times 10^{-8}\{\ln(R)\}^3$$

Use the bisection method of finding roots of equations to find the resistance R at $18.99\,^{\circ}\mathrm{C}$. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

Solving

$$\frac{1}{18.99+273.15}=1.129241\times10^{-3}+2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^{3}$$

we get

$$f(R)=2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^{3}-2.293775\times10^{-3}$$

Lets us assume

$$R_{\ell}=11000,\ R_{u}=14000$$

Check if the function changes sign between R_{ℓ} and R_{u}.

$$\begin{aligned}f(R_{\ell})&=f(11000)\\&=2.341077\times10^{-4}\ln(11000)+8.775468\times10^{-8}\{\ln(11000)\}^{3}-2.293775\times10^{-3}\\&=-4.4536\times10^{-5}\end{aligned}$$

$$\begin{aligned}f(R_{u})&=f(14000)\\&=2.341077\times10^{-4}\ln(14000)+8.775468\times10^{-8}\{\ln(14000)\}^{3}-2.293775\times10^{-3}\\&=1.7563\times10^{-5}\end{aligned}$$

Hence

$$f(R_{\ell})f(R_{u})=f(11000)f(14000)=\left(-4.4536\times10^{-5}\right)\left(1.7563\times10^{-5}\right)<0$$

So there is at least one root between R_{ℓ} and R_{u}, that is, between 11000 and 14000.

<u>Iteration 1</u>

The estimate of the root is

$$\begin{aligned}R_{m}&=\frac{R_{\ell}+R_{u}}{2}\\&=\frac{11000+14000}{2}\\&=12500\end{aligned}$$

$$\begin{aligned}f(R_{m})&=f(12500)\\&=2.341077\times10^{-4}\ln(12500)+8.775468\times10^{-8}\{\ln(12500)\}^{3}-2.293775\times10^{-3}\\&=-1.1655\times10^{-5}\end{aligned}$$

$$f(R_{\ell})f(R_{m})=f(11000)f(12500)=\left(-4.4536\times10^{-5}\right)\left(-1.1655\times10^{-5}\right)>0$$

Hence the root is bracketed between R_{m} and R_{u}, that is, between 12500 and 14000. So, the lower and upper limits of the new bracket are

$$R_{\ell}=12500,\ R_{u}=14000$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$R_m = \frac{R_\ell + R_u}{2}$$
$$= \frac{12500 + 14000}{2}$$
$$= 13250$$

$$f(R_m) = f(13250)$$
$$= 2.341077\times10^{-4}\ln(13250) + 8.775468\times10^{-8}\{\ln(13250)\}^3 - 2.293775\times10^{-3}$$
$$= 3.3599\times10^{-6}$$
$$f(R_\ell)f(R_m) = f(12500)f(13250) = (-1.1655\times10^{-5})(3.3599\times10^{-6}) < 0$$

Hence, the root is bracketed between R_ℓ and R_m, that is, between 12500 and 13250. So the lower and upper limits of the new bracket are

$$R_\ell = 12500,\ R_u = 13250$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{R_m^{\text{new}} - R_m^{\text{old}}}{R_m^{\text{new}}}\right| \times 100$$
$$= \left|\frac{13250 - 12500}{13250}\right| \times 100$$
$$= 5.6604\%$$

None of the significant digits are at least correct in the estimated root

$$R_m = 13250$$

as the absolute relative approximate error is greater than 5%.

Iteration 3

$$R_m = \frac{R_\ell + R_u}{2}$$
$$= \frac{12500 + 13250}{2}$$
$$= 12875$$
$$f(R_m) = f(12875)$$
$$= 2.341077\times10^{-4}\ln(12875) + 8.775468\times10^{-8}\{\ln(12875)\}^3 - 2.293775\times10^{-3}$$
$$= -4.0403\times10^{-6}$$
$$f(R_\ell)f(R_m) = f(12500)f(12875) = ((-1.1654\times10^{-5}))(-4.0398\times10^{-6}) > 0$$

Hence, the root is bracketed between R_m and R_u, that is, between 12875 and 13250.
So, the lower and upper limits of the new bracket are

$$R_\ell = 12875,\ R_u = 13250$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{R_m^{\text{new}} - R_m^{\text{old}}}{R_m^{\text{new}}}\right| \times 100$$

$$= \left|\frac{12875 - 13250}{12875}\right| \times 100$$

$$= 2.9126\%$$

One of the significant digits is at least correct in the estimated root of the equation as the absolute relative approximate error is less than 5%.
Seven more iterations were conducted and these iterations are shown in the Table 1.

Table 1 Root of $f(x) = 0$ as a function of the number of iterations for bisection method.

Iteration	R_ℓ	R_u	R_m	$\lvert\in_a\rvert\%$	$f(R_m)$
1	11000	14000	12500	----------	1.1655×10^{-5}
2	12500	14000	13250	5.6604	3.3599×10^{-6}
3	12500	13250	12875	2.9126	-4.0403×10^{-6}
4	12875	13250	13063	1.4354	-3.1417×10^{-7}
5	13063	13250	13156	0.71259	1.5293×10^{-6}
6	13063	13156	13109	0.35757	6.0917×10^{-7}
7	13063	13109	13086	0.17910	1.4791×10^{-7}
8	13063	13086	13074	0.089633	-8.3022×10^{-8}
9	13074	13086	13080	0.044796	3.2470×10^{-8}
10	13074	13080	13077	0.022403	-2.5270×10^{-8}

At the end of the 10^{th} iteration,

$$|\in_a| = 0.022403\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$|\in_a| \le 0.5\times10^{2-m}$$

$$0.022403 \le 0.5\times10^{2-m}$$

$$0.044806 \le 10^{2-m}$$

$$\log(0.044806) \le 2 - m$$

$$m \le 2 - \log(0.044806) = 3.3487$$

So

$$m = 3$$

The number of significant digits at least correct in the estimated root 13077 is 3.

NONLINEAR EQUATIONS	
Topic	Bisection Method-More Examples
Summary	Examples of Bisection Method
Major	Electrical Engineering
Authors	Autar Kaw
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Web Site	http://numericalmethods.eng.usf.edu