

07.06 Gauss Quadrature Rule for Integration-More Examples Computer Engineering

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to be integrated.

$$I = \int_0^{100} f(x) dx$$

Where,

$$f(x) = 0, \quad 0 < x < 30$$

$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172$$

$$= 0, \quad 172 < x < 200$$

Use two-point Gauss Quadrature Rule to find the value of the integral. Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[0, 100]$ to $[-1, 1]$ using

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

gives

$$\begin{aligned} \int_0^{100} f(x) dx &= \frac{100-0}{2} \int_{-1}^1 f\left(\frac{100-0}{2}x + \frac{100+0}{2}\right) dx \\ &= 50 \int_{-1}^1 f(50x + 50) dx \end{aligned}$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = 1.0000$$

$$x_1 = -0.57735$$

$$c_2 = 1.0000$$

$$x_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$\begin{aligned}
50 \int_{-1}^1 f(50x + 50) dx &\approx 50 [c_1 f(50x_1 + 50) + c_2 f(50x_2 + 50)] \\
&\approx 50 [f(50(-0.57735) + 50) + f(50(0.57735) + 50)] \\
&\approx 50 [f(21.132) + f(78.868)] \\
&\approx 50 [(0) + (0.10492)] \\
&\approx 5.2460
\end{aligned}$$

since

$$\begin{aligned}
f(21.132) &= 0 \\
f(78.868) &= -9.1688 \times 10^{-6} \times (78.868)^3 + 2.7961 \times 10^{-3} \times (78.868)^2 \\
&\quad - 2.8487 \times 10^{-1} \times (78.868) + 9.6778 \\
&= 0.10492
\end{aligned}$$

The absolute relative true error, $|\epsilon_t|$, is (Exact value = 60.793)

$$\begin{aligned}
|\epsilon_t| &= \left| \frac{60.793 - 5.2460}{60.793} \right| \times 100\% \\
&= 91.371\%
\end{aligned}$$

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to be integrated.

$$I = \int_0^{100} f(x) dx$$

where

$$\begin{aligned}
f(x) &= 0, \quad 0 < x < 30 \\
&= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172 \\
&= 0, \quad 172 < x < 200
\end{aligned}$$

Use three-point Gauss Quadrature to find the value of the integral.

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[0, 100]$ to $[-1, 1]$ using

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

gives

$$\begin{aligned}\int_0^{100} f(x)dx &= \frac{100-0}{2} \int_{-1}^1 f\left(\frac{100-0}{2}x + \frac{100+0}{2}\right)dx \\ &= 50 \int_{-1}^1 f(50x+50)dx\end{aligned}$$

The weighting factors and function argument values are

$$c_1 = 0.55556$$

$$x_1 = -0.77460$$

$$c_2 = 0.88889$$

$$x_2 = 0.0000$$

$$c_3 = 0.55556$$

$$x_3 = 0.77460$$

and the formula is

$$\begin{aligned}\int_{-1}^1 f(50x+50)dx &\approx 50[c_1 f(50x_1+50) + c_2 f(50x_2+50) + c_3 f(50x_3+50)] \\ &\approx 50 \left[\begin{array}{l} 0.55556 f(50(-0.77460)+50) \\ + 0.88889 f(50(0.0000)+50) \\ + 0.55556 f(50(0.77460)+50) \end{array} \right] \\ &\approx 50[0.55556 f(11.270) + 0.88889 f(50) + 0.55556 f(88.729)] \\ &\approx 50[0.55556(0) + 0.88889(1.2785) + 0.55556(0.0099462)] \\ &\approx 57.096\end{aligned}$$

since

$$f(11.270) = 0$$

$$\begin{aligned}f(50) &= -9.1688 \times 10^{-6} \times (50)^3 + 2.7961 \times 10^{-3} \times (50)^2 - 2.8487 \times 10^{-1} \times (50) + 9.6778 \\ &= 1.2785\end{aligned}$$

$$\begin{aligned}f(88.730) &= -9.1688 \times 10^{-6} \times (88.729)^3 + 2.7961 \times 10^{-3} \times (88.729)^2 \\ &\quad - 2.8487 \times 10^{-1} \times (88.729) + 9.6778 \\ &= 0.0099462\end{aligned}$$

The absolute relative true error, $|\epsilon_t|$, is (Exact value = 60.793)

$$\begin{aligned}|\epsilon_t| &= \left| \frac{60.793 - 57.096}{60.793} \right| \times 100\% \\ &= 6.0802\%\end{aligned}$$