Chapter 05.04
Lagrange Method of Interpolation – More Examples
Computer Engineering

Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

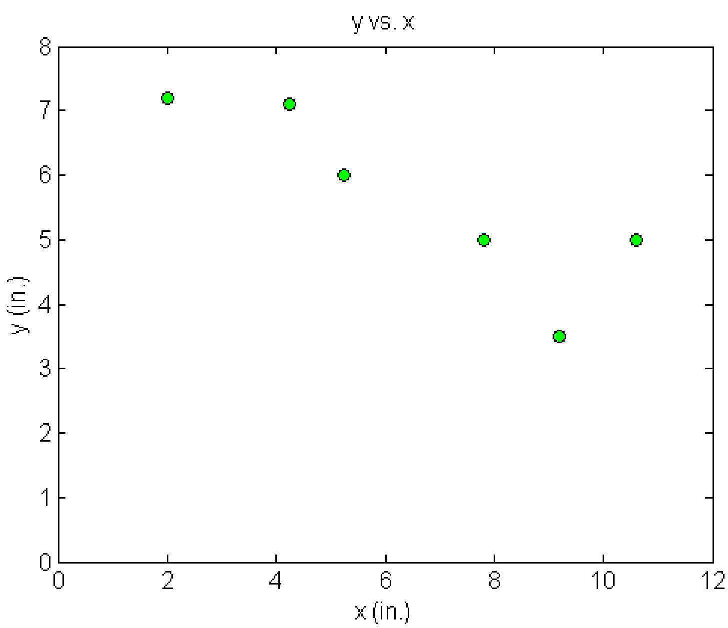

Figure 1 Location of the holes on the rectangular plate.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, what is the value of y at $x = 4.00$ using the Lagrangian method and a first order polynomial?

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), we choose the value of y as given by

$$y(x) = \sum_{i=0}^{1} L_i(x) y(x_i)$$
$$= L_0(x) y(x_0) + L_1(x) y(x_1)$$

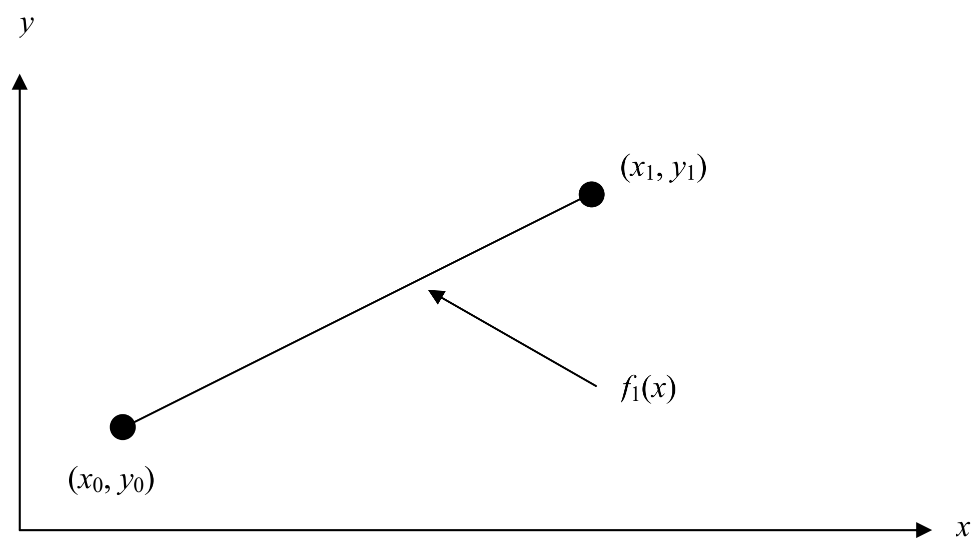

Figure 2 Linear interpolation.

Since we want to find the value of y at $x = 4.00$, using the two points $x_0 = 2.00$ and $x_1 = 4.25$, then

$$x_0 = 2.00,\ y(x_0) = 7.2$$
$$x_1 = 4.25,\ y(x_1) = 7.1$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_0 - x_j}$$
$$= \frac{x - x_1}{x_0 - x_1}$$
$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{x - x_j}{x_1 - x_j}$$

$$= \frac{x - x_0}{x_1 - x_0}$$

Hence

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1)$$

$$= \frac{x - 4.25}{2.00 - 4.25}(7.2) + \frac{x - 2.00}{4.25 - 2.00}(7.1), \ \ 2.00 \le x \le 4.25$$

$$y(4.00) = \frac{4.00 - 4.25}{2.00 - 4.25}(7.2) + \frac{4.00 - 2.00}{4.25 - 2.00}(7.1)$$

$$= 0.11111(7.2) + 0.88889(7.1)$$

$$= 7.1111 \text{ in.}$$

You can see that $L_0(x) = 0.11111$ and $L_1(x) = 0.88889$ are like weightages given to the values of y at $x_0 = 2.00$ and $x_1 = 4.25$ to calculate the value of y at $x = 4.00$.

Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

Table 2 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from $x = 2.00$ to $x = 4.25$ to $x = 5.25$ in a quadratic path, what is the value of y at $x = 4.00$ using a second order Lagrange polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = \sum_{i=0}^{2} L_i(x) y(x_i)$$

$$= L_0(x) y(x_0) + L_1(x) y(x_1) + L_2(x) y(x_2)$$

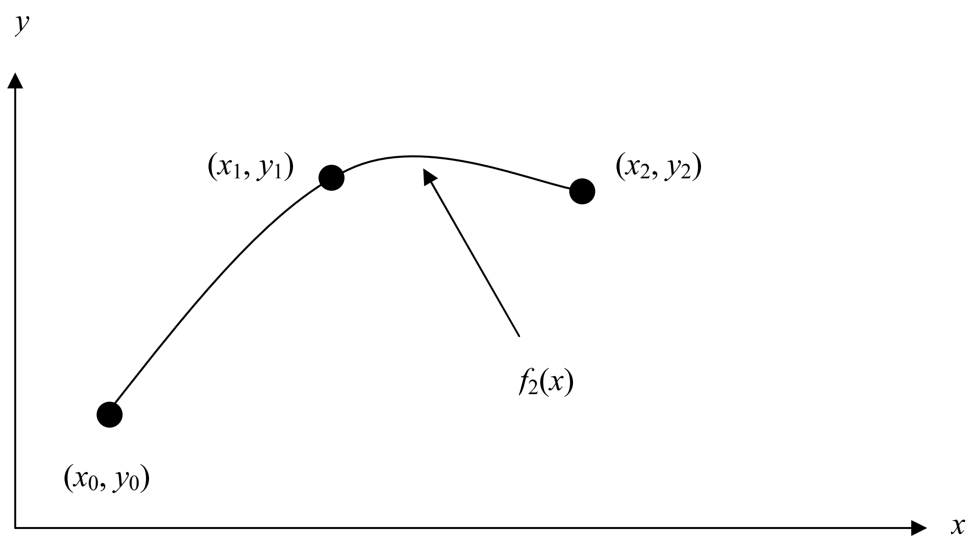

Figure 3 Quadratic interpolation.

Since we want to find the value of y at $x = 4.00$, using the three points as $x_0 = 2.00$, $x_1 = 4.25$ and $x_2 = 5.25$, then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{x - x_j}{x_2 - x_j}$$
$$= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Hence

$$y(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)y(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)y(x_1) + \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)y(x_2),$$

$$x_0 \le x \le x_2$$

$$y(4.00) = \frac{(4.00-4.25)(4.00-5.25)}{(2.00-4.25)(2.00-5.25)}(7.2) + \frac{(4.00-2.00)(4.00-5.25)}{(4.25-2.00)(4.25-5.25)}(7.1)$$

$$+\frac{(4.00-2.00)(4.00-4.25)}{(5.25-2.00)(5.25-4.25)}(6.0)$$

$$= (0.042735)(7.2) + (1.1111)(7.1) + (-0.15385)(6.0)$$

$$= 7.2735 \text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{7.2735-7.1111}{7.2735}\right| \times 100$$

$$= 2.2327\%$$

Example 3

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

Table 3 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Find the path traversed through the six points using a fifth order Lagrange polynomial.

Solution

For fifth order Lagrange polynomial interpolation (also called quintic interpolation), we choose the value of y given by

$$y(x) = \sum_{i=0}^{5} L_i(x) y(x_i)$$

$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2)$$

$$+ L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5)$$

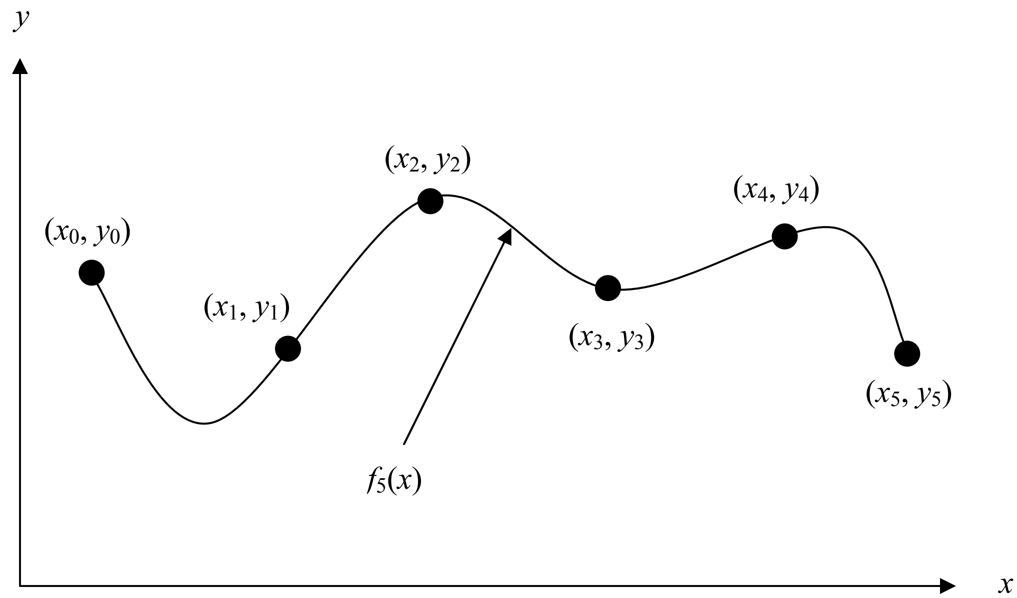

Figure 4 5th order polynomial interpolation.

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$
$$x_3 = 7.81, \quad y(x_3) = 5.0$$
$$x_4 = 9.20, \quad y(x_4) = 3.5$$
$$x_5 = 10.60, \quad y(x_5) = 5.0$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{5} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right)\left(\frac{x - x_4}{x_0 - x_4}\right)\left(\frac{x - x_5}{x_0 - x_5}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{5} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)\left(\frac{x - x_4}{x_1 - x_4}\right)\left(\frac{x - x_5}{x_1 - x_5}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{5} \frac{x - x_j}{x_2 - x_j}$$

$$= \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j\neq 3}}^{5} \frac{x-x_j}{x_3-x_j}$$

$$= \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)$$

$$L_4(x) = \prod_{\substack{j=0 \\ j\neq 4}}^{5} \frac{x-x_j}{x_4-x_j}$$

$$= \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)$$

$$L_5(x) = \prod_{\substack{j=0 \\ j\neq 5}}^{5} \frac{x-x_j}{x_5-x_j}$$

$$= \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)$$

$$y(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)y(x_0)$$

$$+ \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)y(x_1)$$

$$+ \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)y(x_2)$$

$$+ \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)y(x_3)$$

$$+ \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)y(x_4)$$

$$+ \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)y(x_5)$$

$$
\begin{aligned}
&= \frac{(x-4.25)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(2.00-4.25)(2.00-5.25)(2.00-7.81)(2.00-9.20)(2.00-10.60)}(7.2) \\
&+ \frac{(x-2.00)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(4.25-2.00)(4.25-5.25)(4.25-7.81)(4.25-9.20)(4.25-10.60)}(7.1) \\
&+ \frac{(x-2.00)(x-4.25)(x-7.81)(x-9.20)(x-10.60)}{(5.25-2.00)(5.25-4.25)(5.25-7.81)(5.25-9.20)(5.25-10.60)}(6.0) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-9.20)(x-10.60)}{(7.81-2.00)(7.81-4.25)(7.81-5.25)(7.81-9.20)(7.81-10.60)}(5.0) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-10.60)}{(9.20-2.00)(9.20-4.25)(9.20-5.25)(9.20-7.81)(9.20-10.60)}(3.5) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-9.20)}{(10.60-2.00)(10.60-4.25)(10.60-5.25)(10.60-7.81)(10.60-9.20)}(5.0) \\
&= \frac{x^5-37.11x^4+536.77x^3-3773.2x^2+12862x-16994}{-365.38} \\
&+ \frac{x^5-34.86x^4+462.83x^3-2879.7x^2+8169.5x-7997.1}{35.461} \\
&+ \frac{x^5-33.86x^4+433.22x^3-2572.3x^2+6903.5x-6473.9}{-29.304} \\
&+ \frac{x^5-31.3x^4+366.53x^3-1984.1x^2+4912.4x-4351.8}{41.069} \\
&+ \frac{x^5-29.91x^4+335.81x^3-1757.2x^2+4241.6x-3694.3}{-78.273} \\
&+ \frac{x^5-28.51x^4+308.78x^3-1573.7x^2+3727.5x-3206.4}{228.24}
\end{aligned}
$$

$$
y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \le x \le 10.6
$$

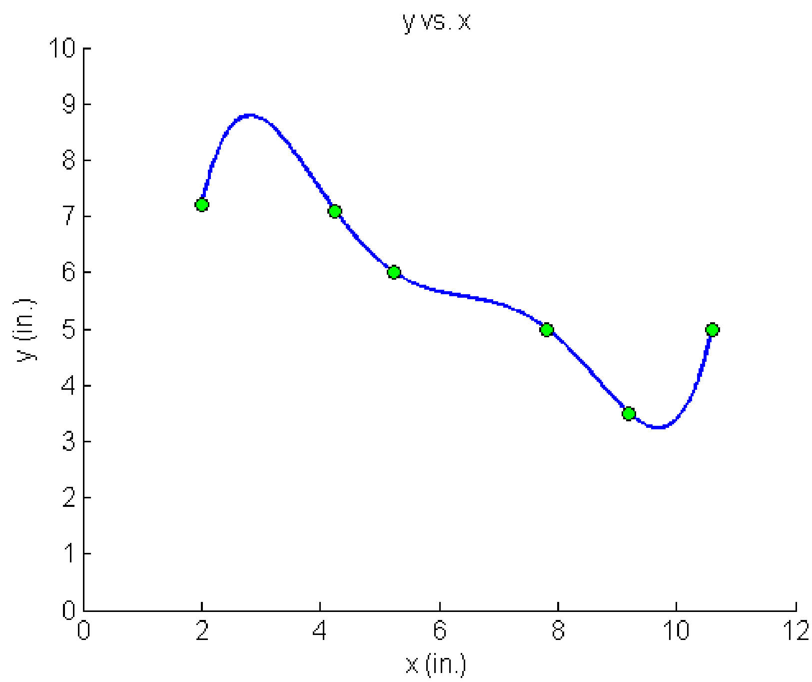

Figure 5 Fifth order polynomial to traverse points of robot path (using Lagrangian method of interpolation).