Chapter 02.02
Differentiation of Continuous Functions-More Examples
Computer Engineering

Example 1

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use the forward divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.

b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

a) $$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$\begin{aligned} x_{i+1} &= x + \Delta x \\ &= 0.1 + 0.05 \\ &= 0.15 \end{aligned}$$

$$\begin{aligned} f(0.1) &= 1 - e^{(-0.24 \times 0.1)} \\ &= 0.023714 \end{aligned}$$

$$\begin{aligned} f(0.15) &= 1 - e^{(-0.24 \times 0.15)} \\ &= 0.035360 \end{aligned}$$

$$f'(0.1) \approx \frac{f(0.15) - f(0.1)}{0.05}$$
$$= \frac{0.035360 - 0.023714}{0.05}$$
$$= 0.23291$$

The exact value of $f'(0.1)$ can be calculated by differentiating

$$f(x) = 1 - e^{-ax},\ x \geq 0$$

as

$$f'(x) = \frac{d}{dx}[f(x)]$$

Knowing that

$$\frac{d}{dx}\left[e^{-ax}\right] = -ae^{-ax}$$

we find

$$f'(x) = \frac{d}{dx}(1 - e^{-ax})$$
$$= ae^{-ax}$$
$$= 0.24e^{-0.24x}$$
$$f'(0.1) = (0.24)\left(e^{-(0.24 \times 0.1)}\right)$$
$$= 0.23431$$

The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{0.23431 - 0.23291}{0.23431}\right| \times 100$$
$$= 0.59761\%$$

b) $a = 0.12$

$$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$$
$$= 0.011928$$
$$f(0.15) = 1 - e^{-(0.12 \times 0.15)}$$
$$= 0.017839$$
$$f'(0.1) \approx \frac{f(0.15) - f(0.1)}{0.05}$$
$$= \frac{0.017839 - 0.011928}{0.05}$$
$$= 0.11821$$
$$f'(x) = \frac{d}{dx}(1 - e^{-ax})$$
$$= ae^{-ax}$$
$$= 0.12e^{-0.12x}$$

$$f'(0.1) = (0.12)(e^{-(0.12\times 0.1)}$$
$$= 0.11856$$

The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{0.11857 - 0.11821}{0.11857}\right| \times 100$$
$$= 0.29940\%$$

The estimate of the derivative decreased.

Example 2

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use the backward divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.
b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

a) $$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$a = 0.24$$
$$x_i = 0.1$$
$$\Delta x = 0.05$$
$$x_{i-1} = x_i - \Delta x$$
$$= 0.1 - 0.05$$
$$= 0.05$$
$$f(0.1) = 1 - e^{-(0.24\times 0.1)}$$
$$= 0.023714$$
$$f(0.05) = 1 - e^{-(0.24\times 0.05)}$$
$$= 0.011928$$
$$f'(0.1) \approx \frac{f(0.1) - f(0.05)}{0.05}$$

$$= \frac{0.023714 - 0.011928}{0.05}$$
$$= 0.23572$$

The absolute relative true error is

$$|\in_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$
$$= \left| \frac{0.23431 - 0.23572}{0.23431} \right| \times 100$$
$$= 0.60241\%$$

b) $a = 0.12$

$$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$$
$$= 0.011928$$
$$f(0.05) = 1 - e^{-(0.12 \times 0.05)}$$
$$= 0.0059820$$
$$f'(0.1) \approx \frac{f(0.1) - f(0.05)}{0.05}$$
$$= \frac{0.011928 - 0.0059820}{0.05}$$
$$= 0.11893$$

The absolute relative true error is

$$|\in_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$
$$= \left| \frac{0.11857 - 0.11893}{0.11857} \right| \times 100$$
$$= 0.30060\%$$

The estimate of the derivative decreased.

Example 3

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use the central divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.
b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

a) $$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x}$$

$$a = 0.24$$
$$x_i = 0.1$$
$$\Delta x = 0.05$$
$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ &= 0.1 + 0.05 \\ &= 0.15 \end{aligned}$$
$$\begin{aligned} x_{i-1} &= x_i - \Delta x \\ &= 0.1 - 0.05 \\ &= 0.05 \end{aligned}$$
$$\begin{aligned} f'(0.1) &\approx \frac{f(0.15) - f(0.05)}{2(0.05)} \\ &= \frac{f(0.15) - f(0.05)}{0.1} \end{aligned}$$
$$\begin{aligned} f(0.15) &= 1 - e^{-(0.24\times 0.15)} \\ &= 0.035360 \end{aligned}$$
$$\begin{aligned} f(0.05) &= 1 - e^{-(0.24\times 0.05)} \\ &= 0.011928 \end{aligned}$$

$$\begin{aligned} f'(0.1) &\approx \frac{f(0.15) - f(0.05)}{0.1} \\ &= \frac{0.035360 - 0.011928}{0.1} \\ &= 0.23431 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned} |\in_t| &= \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100 \\ &= \left|\frac{0.23431 - 0.23431}{0.23431}\right| \times 100 \\ &= 0.0024000\% \end{aligned}$$

b) $$a = 0.12$$
$$\begin{aligned} f(0.15) &= 1 - e^{-(0.12\times 0.15)} \\ &= 0.017839 \end{aligned}$$
$$\begin{aligned} f(0.05) &= 1 - e^{-(0.12\times 0.05)} \\ &= 0.0059820 \end{aligned}$$

$$f'(0.1) \approx \frac{f(0.15) - f(0.05)}{0.1}$$

$$= \frac{0.017839 - 0.0059820}{0.1}$$
$$= 0.11857$$

The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{0.11857 - 0.11857}{0.11857}\right| \times 100$$
$$= 6.0000 \times 10^{-4}\%$$

The results from the three difference approximations in Examples 1, 2 and 3 are given in Table 1.

Table 1 Summary of $f'(0.1)$ values using different divided difference approximations.

Type of Difference Approximation	$f'(0.1), a = 0.24$	$\|\in_t\|\%, a = 0.24$	$f'(0.1), a = 0.12$	$\|\in_t\|\%, a = 0.12$
Forward	0.23291	0.59761	0.11821	0.29940
Backward	0.23572	0.60241	0.11893	0.30060
Central	0.23431	0.0024000	0.11857	6.0000×10^{-4}

Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size.
In real life, one would not know the exact value of the derivative, so how would one know how accurately they have found the value of the derivative?
A simple way would be to start with a step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.
Take the example of finding $f'(x)$ for $f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$ at $x = 0.1$ for $a = 0.24$ using the backward divided difference scheme. Given in Table 2 are the values obtained using the backward divided difference approximation method and the corresponding absolute relative approximate errors.

Table 2 Values of $f'(0.1)$ using backward different divided difference approximation with different step sizes.

Δx	$f'(x)$	$\|\in_a\|\%$
0.05	0.23572	
0.025	0.23501	0.30090
0.0125	0.23466	0.15023
0.00625	0.23448	0.075056
0.003125	0.23440	0.037514

DIFFERENTIATION	
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