Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Civil Engineering

Example 1

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0,\ C(0)=10^7$$

Using the Runge-Kutta 4$^{\text{th}}$ order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

Solution

$$\frac{dC}{dt}=-0.06C$$

$$f(t,C)=-0.06C$$

$$C_{i+1}=C_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$

For $i=0$, $t_0=0$, $C_0=10^7$

$$\begin{aligned}k_1&=f(t_0,C_0)\\&=f\left(0,10^7\right)\\&=-0.06\left(10^7\right)\\&=-600000\end{aligned}$$

$$\begin{aligned}k_2&=f\left(t_0+\frac{1}{2}h,C_0+\frac{1}{2}k_1h\right)\\&=f\left(0+\frac{1}{2}\times3.5,10^7+\frac{1}{2}(-600000)3.5\right)\\&=f(1.75,8950000)\\&=-0.06(8950000)\\&=-537000\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_2 h\right)\\
&= f\left(0 + \frac{1}{2}3.5, 10^7 + \frac{1}{2}(-537000)3.5\right)\\
&= f(1.75, 9060300)\\
&= -0.06(9060300)\\
&= -543620\\
k_4 &= f(t_0 + h, C_0 + k_3 h)\\
&= f(0 + 3.5, 10^7 + (-543620)3.5)\\
&= f(3.5, 8097300)\\
&= -0.06(8097300)\\
&= -485840\\
C_1 &= C_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h\\
&= 10^7 + \frac{1}{6}(-600000 + 2(-537000) + 2(-543620) + (-485840))3.5\\
&= 10^7 + \frac{1}{6}(-3247100)3.5\\
&= 8.1059 \times 10^6 \text{ parts/m}^3
\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ parts/m}^3$$

$$C(3.5) \approx C_1 = 8.1059 \times 10^6 \text{ parts/m}^3$$

For $i = 1, t_1 = 3.5, C_1 = 8.1059 \times 10^6$

$$\begin{aligned}
k_1 &= f(t_1, C_1)\\
&= f(3.5, 8.1059 \times 10^6)\\
&= -0.06(8.1059 \times 10^6)\\
&= -486350\\
k_2 &= f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_1 h\right)\\
&= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-486350)3.5\right)\\
&= f(5.25, 7254800)\\
&= -0.06(7254800)\\
&= -435290\\
k_3 &= f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_2 h\right)\\
&= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-435290)3.5\right)
\end{aligned}$$

$$
\begin{aligned}
&= f(5.25, 7344100) \\
&= -0.06(7344100) \\
&= -440648 \\
k_4 &= f(t_1 + h, C_1 + k_3 h) \\
&= f(3.5 + 3.5, 8105900 + (-440648)3.5) \\
&= f(7, 6563600) \\
&= -0.06(6563600) \\
&= -393820 \\
C_2 &= C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 8105900 + \frac{1}{6}(-486350 + 2\times(-435290) + 2\times(-440648) + (-393820))\times 3.5 \\
&= 8105900 + \frac{1}{6}(-2632000)\times 3.5 \\
&= 6.5705\times 10^6 \text{ parts/m}^3
\end{aligned}
$$

C_2 is the approximate concentration of bacteria at

$$t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C(7) \approx C_2 = 6.5705\times 10^6 \text{ parts/m}^3$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1\times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705\times 10^6 \text{ parts/m}^3$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

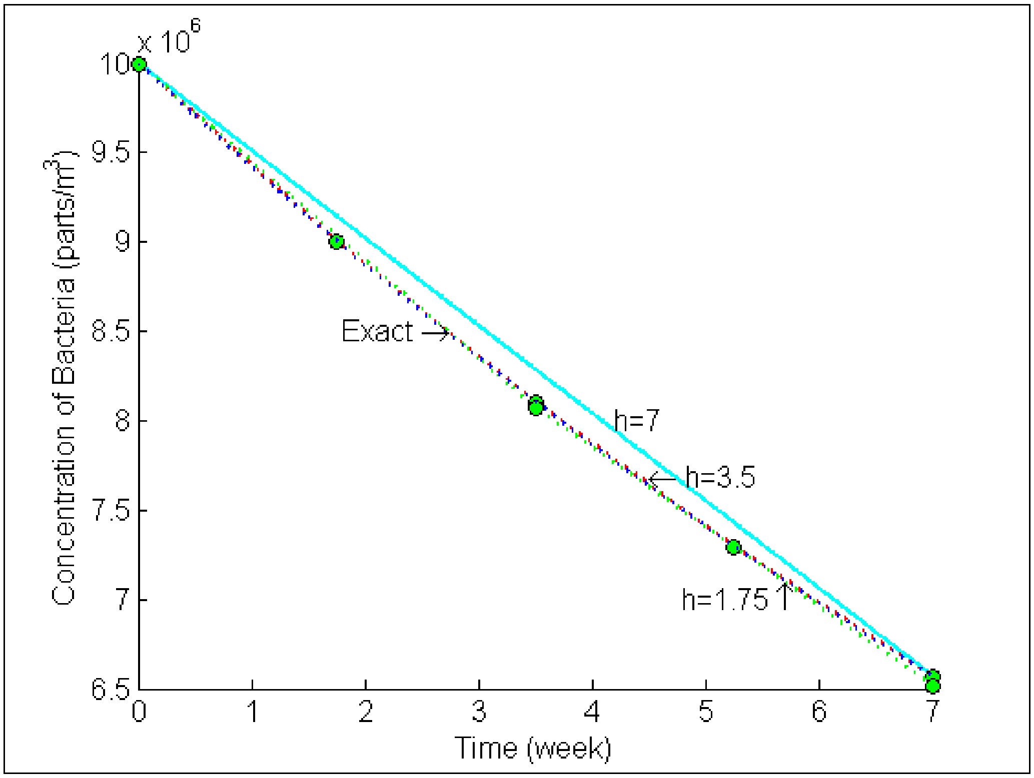

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated concentration of bacteria at $t = 7$ weeks.

Table 1 Value of concentration of bacteria at 7 weeks for different step sizes.

Step size, h	$C(7)$	E_t	$\lvert\in_t\rvert$ %
7	6.5715×10^6	-1017.2	0.015481
3.5	6.5705×10^6	-53.301	8.1121×10^{-4}
1.75	6.5705×10^6	-3.0512	4.6438×10^{-5}
0.875	6.5705×10^6	-0.18252	2.7779×10^{-6}
0.4375	6.5705×10^6	-0.011161	1.6986×10^{-7}

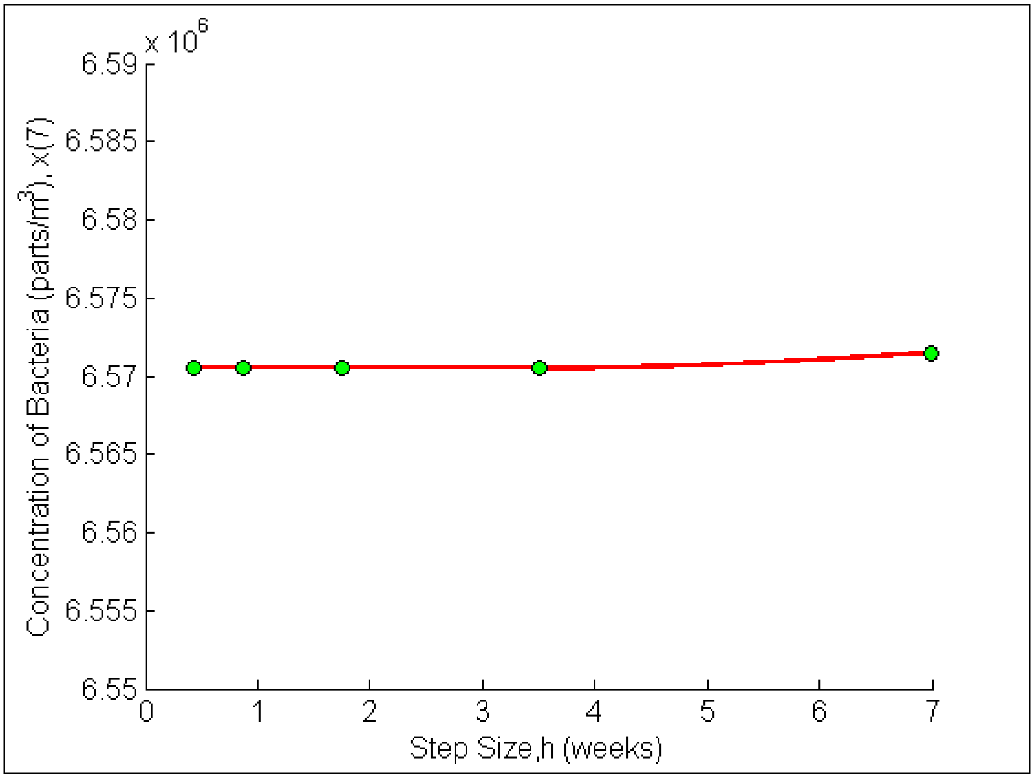

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and the Runge-Kutta 4th order method.

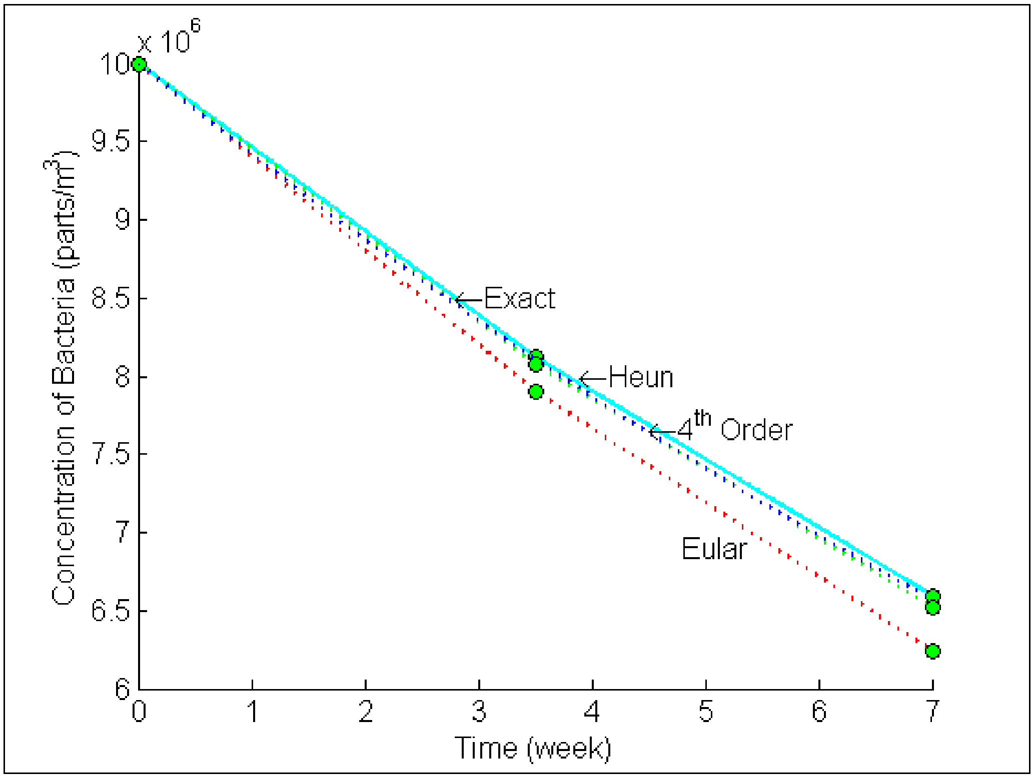

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.