Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Civil Engineering

Example 1
A polluted lake has an initial concentration of a bacteria of $10^7$ parts/m$^3$, while the acceptable level is only $5 \times 10^6$ parts/m$^3$. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration $C$ of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, \ C(0) = 10^7$$

Using Runge-Kutta 2nd order method and a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks.

Solution

$$\frac{dC}{dt} = -0.06C$$
$$f(t, C) = -0.06C$$

Per Heun’s method

$$C_{i+1} = C_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h$$

$$k_1 = f(t_i, C_i)$$
$$k_2 = f(t_i + h, C_i + k_i h)$$

For $i = 0, \ t_0 = 0, \ C_0 = 10^7$

$$k_1 = f(t_0, C_0)$$
$$= f(0, 10^7)$$
$$= -0.06(10^7)$$
$$= -600000$$

$$k_2 = f(t_0 + h, C_0 + k_i h)$$
$$= f(0 + 3.5, 10^7 + (-600000)3.5)$$
$$= f(3.5, 7.9 \times 10^6)$$
$$= -0.06(7.9 \times 10^6)$$
\[ C_1 = C_0 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \]
\[ = 10^7 + \left( \frac{1}{2} (-600000) + \frac{1}{2} (-474000) \right) 3.5 \]
\[ = 10^7 + (-537000) 3.5 \]
\[ = 8.1205 \times 10^6 \text{ parts/m}^3 \]

\( C_1 \) is the approximate concentration of bacteria at

\[ t = t_i = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks} \]
\[ C(3.5) \approx C_1 = 8.1205 \times 10^6 \text{ parts/m}^3 \]

For \( i = 1, t_i = t_0 + h = 0 + 3.5 = 3.5, C_1 = 8.1205 \times 10^6 \)

\[ k_1 = f(t_i, C_i) \]
\[ = f(3.5, 8.1205 \times 10^6) \]
\[ = -0.06(8.1205 \times 10^6) \]
\[ = -487230 \]

\[ k_2 = f(t_i + h, C_i + k_1 h) \]
\[ = f(3.5 + 3.5, 8.1205 \times 10^6 + (-487230)3.5) \]
\[ = f(7, 6415200) \]
\[ = -0.06(6415200) \]
\[ = -384910 \]

\[ C_2 = C_1 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \]
\[ = 8.1205 \times 10^6 + \left( \frac{1}{2} (-487230) + \frac{1}{2} (-384910) \right) 3.5 \]
\[ = 8.1205 \times 10^6 + (-436070) 3.5 \]
\[ = 6.5943 \times 10^6 \text{ parts/m}^3 \]

\( C_2 \) is the approximate concentration of bacteria at

\[ t = t_2 = t_i + h = 3.5 + 3.5 = 7 \text{ weeks} \]
\[ C(7) \approx C_2 = 6.5943 \times 10^6 \text{ parts/m}^3 \]

The exact solution of the ordinary differential equation is given by

\[ C(t) = 1 \times 10^7 e^{-\frac{3t}{50}} \]

The solution to this nonlinear equation at \( t = 7 \) weeks is

\[ C(7) = 6.5705 \times 10^6 \text{ parts/m}^3 \]

The results from Heun’s method are compared with exact results in Figure 1.
Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2.

**Table 1** Effect of step size for Heun’s method.

| Step size, $h$ | $C(7)$         | $E_t$   | $|\varepsilon_t|\%$ |
|---------------|----------------|---------|-------------------|
| 7             | $6.6820 \times 10^6$ | $-111530$ | 1.6975           |
| 3.5           | $6.5943 \times 10^6$ | $-23784$  | 0.36198          |
| 1.75          | $6.5760 \times 10^6$ | $-5489.1$ | 0.083542         |
| 0.875         | $6.5718 \times 10^6$ | $-1318.8$ | 0.020071         |
| 0.4375        | $6.5708 \times 10^6$ | $-323.24$ | 0.0049195        |
In Table 2, the Euler’s method and Runge-Kutta 2nd order method results are shown as a function of step size.

<table>
<thead>
<tr>
<th>Step size, $h$</th>
<th>Euler</th>
<th>Heun</th>
<th>Midpoint</th>
<th>Ralston</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$5.8000 \times 10^6$</td>
<td>$6.6820 \times 10^6$</td>
<td>$6.6820 \times 10^6$</td>
<td>$6.6820 \times 10^6$</td>
</tr>
<tr>
<td>3.5</td>
<td>$6.2410 \times 10^6$</td>
<td>$6.5943 \times 10^6$</td>
<td>$6.5943 \times 10^6$</td>
<td>$6.5943 \times 10^6$</td>
</tr>
<tr>
<td>1.75</td>
<td>$6.4160 \times 10^6$</td>
<td>$6.5760 \times 10^6$</td>
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</tr>
<tr>
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<td>$6.5340 \times 10^6$</td>
<td>$6.5708 \times 10^6$</td>
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<td>$6.5708 \times 10^6$</td>
</tr>
</tbody>
</table>

While in Figure 3, the comparison is shown over the range of time.
Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.