Chapter 08.02
Euler's Method for Ordinary Differential Equations-More Examples
Civil Engineering

Example 1

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0,\ C(0)=10^7$$

Using Euler's method and a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks.

Solution

$$\frac{dC}{dt}=-0.06C$$

$$f(t,C)=-0.06C$$

The Euler's method reduces to

$$C_{i+1}=C_i+f(t_i,C_i)h$$

For $i=0$, $t_0=0$, $C_0=10^7$

$$\begin{aligned}C_1&=C_0+f(t_0,C_0)h\\&=10^7+f(0,10^7)3.5\\&=10^7+\left(-0.06(10^7)\right)3.5\\&=10^7+\left(-6\times10^5\right)3.5\\&=7.9\times10^6\text{ parts/m}^3\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t=t_1=t_0+h=0+3.5=3.5\text{ weeks}$$

$$C(3.5)\approx C_1=7.9\times10^6\text{ parts/m}^3$$

For $i=1$, $t_1=3.5$, $C_1=7.9\times10^6$

$$\begin{aligned}C_2&=C_1+f(t_1,C_1)h\\&=7.9\times10^6+f\left(3.5,7.9\times10^6\right)3.5\end{aligned}$$

$$= 7.9\times10^{6} + \left(-0.06(7.9\times10^{6})\right)3.5$$
$$= 7.9\times10^{6} + \left(-4.74\times10^{5}\right)3.5$$
$$= 6.241\times10^{6} \text{ parts/m}^{3}$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$
$$C(7) \approx C_2 = 6.241\times10^{6} \text{ parts/m}^{3}$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1\times10^{7} e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705\times10^{6} \text{ parts/m}^{3}$$

Figure 1 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 3.5$.

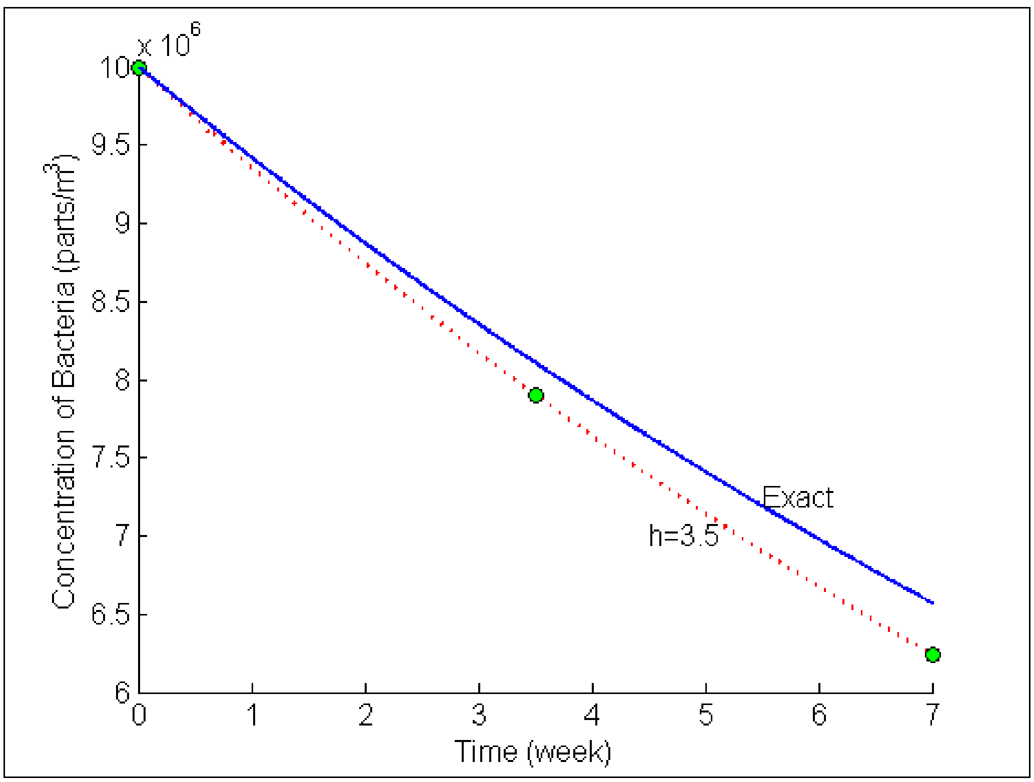

Figure 1 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Concentration of bacteria after 7 weeks as a function of step size, h.

step size, h	$C(7)$	E_t	$\lvert\epsilon_t\rvert\%$
7	5.8×10^{6}	770470	11.726
3.5	6.241×10^{6}	329470	5.0144
1.75	6.4164×10^{6}	154060	2.3447
0.875	6.4959×10^{6}	74652	1.1362
0.4375	6.5337×10^{6}	36763	0.55952

Figure 2 shows how the concentration of bacteria varies as a function of time for different step sizes.

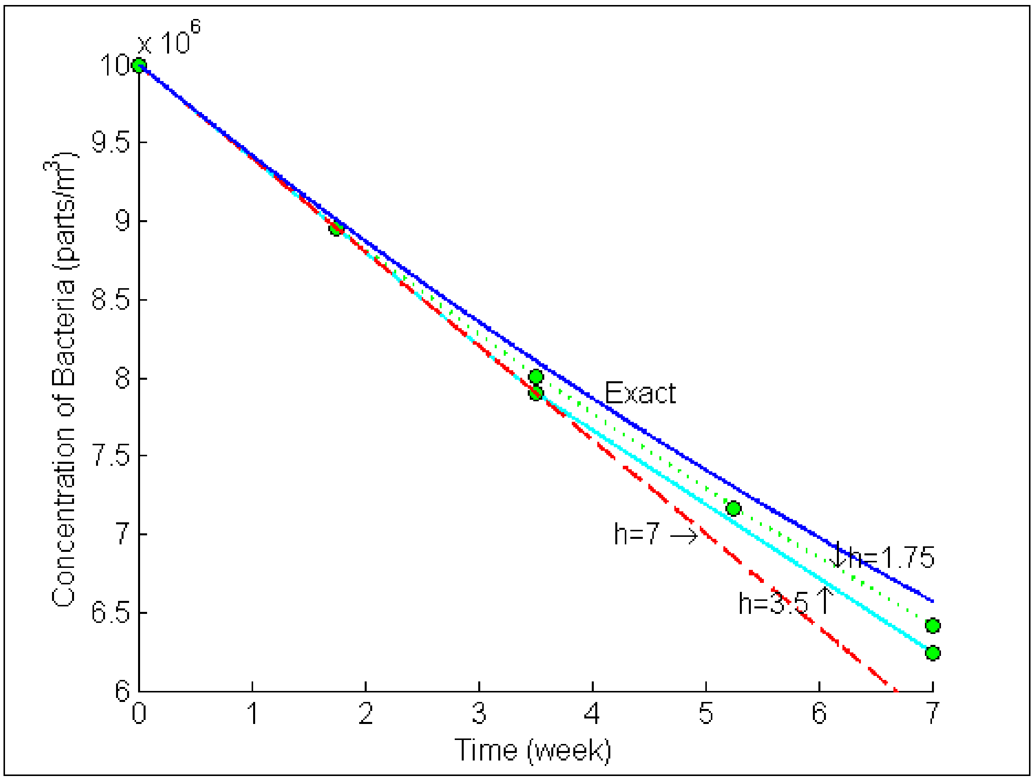

Figure 2 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated concentration of bacteria at $t = 7$ weeks as a function of step size are plotted in Figure 3.

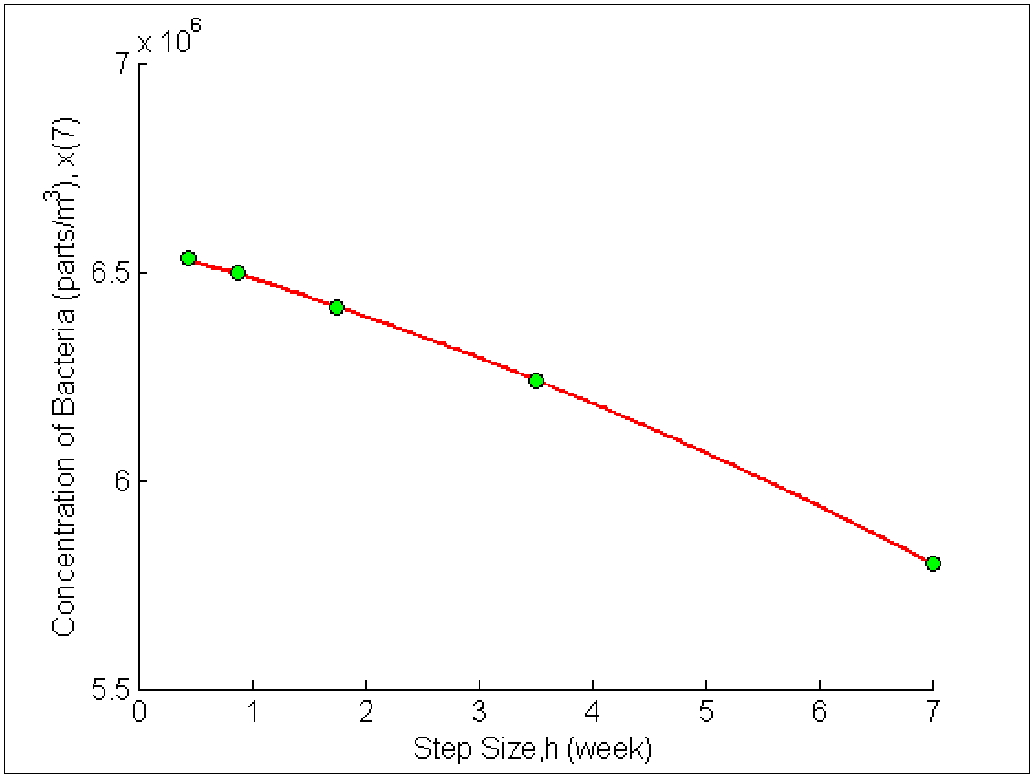

Figure 3 Effect of step size in Euler's method.