Chapter 08.02
Euler’s Method for Ordinary Differential Equations-
More Examples
Civil Engineering

Example 1
A polluted lake has an initial concentration of a bacteria of $10^7$ parts/m$^3$, while the acceptable level is only $5 \times 10^6$ parts/m$^3$. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration $C$ of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, \quad C(0) = 10^7$$

Using Euler’s method and a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks.

Solution

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

The Euler’s method reduces to

$$C_{i+1} = C_i + f(t_i, C_i)h$$

For $i = 0, \ t_0 = 0, \ C_0 = 10^7$

$$C_1 = C_0 + f(t_0, C_0)h$$

$$= 10^7 + f(0, 10^7) \cdot 3.5$$

$$= 10^7 + (-0.06 \cdot 10^7) \cdot 3.5$$

$$= 10^7 - 6 \times 10^5 \cdot 3.5$$

$$= 7.9 \times 10^6 \text{ parts/m}^3$$

$C_1$ is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks}$$

$$C(3.5) \approx C_1 = 7.9 \times 10^6 \text{ parts/m}^3$$

For $i = 1, \ t_1 = 3.5, \ C_1 = 7.9 \times 10^6$

$$C_2 = C_1 + f(t_1, C_1)h$$

$$= 7.9 \times 10^6 + f(3.5, 7.9 \times 10^6) \cdot 3.5$$
\[
\begin{align*}
&= 7.9 \times 10^6 + (-0.06(7.9 \times 10^6)) \times 3.5 \\
&= 7.9 \times 10^6 + (-4.74 \times 10^5) \times 3.5 \\
&= 6.241 \times 10^6 \text{ parts/m}^3
\end{align*}
\]

\( C_2 \) is the approximate concentration of bacteria at
\[ t = t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks} \]
\[ C(7) \approx C_2 = 6.241 \times 10^6 \text{ parts/m}^3 \]

The exact solution of the ordinary differential equation is given by
\[ C(t) = 1 \times 10^7 e^{\left(-\frac{3t}{50}\right)} \]

The solution to this nonlinear equation at \( t = 7 \) weeks is
\[ C(7) = 6.5705 \times 10^6 \text{ parts/m}^3 \]

Figure 1 compares the exact solution with the numerical solution from Euler’s method for the step size of \( h = 3.5 \).

![Figure 1 Comparing exact and Euler’s method.](image-url)

The problem was solved again using smaller step sizes. The results are given below in Table 1.
Table 1  Concentration of bacteria after 7 weeks as a function of step size, $h$.

| step size, $h$ | $C(7)$   | $E_t$    | $|\varepsilon|\%$ |
|---------------|----------|----------|------------------|
| 7             | $5.8 \times 10^6$ | 770470  | 11.726           |
| 3.5           | $6.241 \times 10^6$ | 329470  | 5.0144           |
| 1.75          | $6.4164 \times 10^6$ | 154060  | 2.3447           |
| 0.875         | $6.4959 \times 10^6$ | 74652   | 1.1362           |
| 0.4375        | $6.5337 \times 10^6$ | 36763   | 0.55952          |

Figure 2 shows how the concentration of bacteria varies as a function of time for different step sizes.

While the values of the calculated concentration of bacteria at $t = 7$ weeks as a function of step size are plotted in Figure 3.
Figure 3 Effect of step size in Euler’s method.