07.05
Romberg Rule for Integration-More Examples
Civil Engineering

Example 1

The concentration of benzene at a critical location is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73} erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2} dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

Table 1 Values obtained for Trapezoidal rule for

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

n	Trapezoidal Rule
1	-1.4124
2	-0.70695
4	-0.40571
8	-0.33475

a) Use Romberg's rule to find $erfc(0.6560)$. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $I_2 = -0.70695$

$I_4 = -0.40571$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$TV \approx I_4 + \frac{I_4 - I_2}{3}$$

$$\approx -0.40571 + \frac{-0.40571 - (-0.70695)}{3}$$

$$\approx -0.30530$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value for calculating the true error and relative true error.

$$erfc(0.6560) = \int_5^{0.6560} e^{-z^2}\, dz$$

$$= -0.31333$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$= -0.31333 - (-0.30530)$$

$$= -0.0080295$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$

$$= \left|\frac{-0.0080295}{-0.31333}\right| \times 100\,\%$$

$$= 2.5627\,\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Table 2 Values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$erfc(0.6560) = \int_5^{0.6560} e^{-z^2}\, dz$$

n	Trapezoidal Rule	$\lvert\in_t\rvert$ for Trapezoidal Rule %	Richardson's Extrapolation	$\lvert\in_t\rvert$ for Richardson's Extrapolation %
1	−1.4124	350.79	--	--
2	−0.70695	125.63	−0.47180	50.578
4	−0.40571	29.483	−0.30530	2.5627
8	−0.33475	6.8383	−0.31110	0.71156

Example 2

The concentration of benzene at a critical location is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73} erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2} dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

Use Romberg's rule to find $erfc(0.6560)$. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = -1.4124$$

$$I_{1,2} = -0.70695$$

$$I_{1,3} = -0.40571$$

$$I_{1,4} = -0.33475$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= -0.70695 + \frac{-0.70695 - (-1.4124)}{3} \\ &= -0.47180 \end{aligned}$$

Similarly

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= -0.40571 + \frac{-0.40571 - (-0.70695)}{3} \\ &= -0.30530 \end{aligned}$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= -0.33475 + \frac{-0.33475-(-0.40571)}{3}$$
$$= -0.31110$$

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$
$$= -0.30530 + \frac{-0.30530-(-0.47180)}{15}$$
$$= -0.29420$$

Similarly

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$
$$= -0.31110 + \frac{-0.31110-(-0.30530)}{15}$$
$$= -0.31148$$

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$
$$= -0.31148 + \frac{-0.31148-(-0.29420)}{63}$$
$$= -0.31176$$

Table 2 shows these increased correct values in a tree graph.

Table 3 Improved estimates of value of integral using Romberg integration.

		1st Order	2nd Order	3rd Order
1-segment	−1.4124			
		−0.47180		
			−0.29420	
2-segment	−0.70695			
				−0.31176
		−0.30530		
			−0.31148	
4-segment	−0.40571			
		−0.31110		
8-segment	−0.33475			