Chapter 04.08
Gauss-Seidel – More Examples
Civil Engineering

Example 1

To find the maximum stresses in a compound cylinder, the following four simultaneous linear equations need to be solved.

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 4.2857\times10^7 & -5.4619\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

In the compound cylinder, the inner cylinder has an internal radius of $a = 5"$, and an outer radius $c = 6.5"$, while the outer cylinder has an internal radius of $c = 6.5"$ and an outer radius of $b = 8"$. Given $E = 30\times10^6$ psi, $\nu = 0.3$, and that the hoop stress in the outer cylinder is given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right],$$

find the stress on the inside radius of the outer cylinder.

Find the values of c_1, c_2, c_3 and c_4 using the Gauss-Seidel Method. Use

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -0.005 \\ 0.001 \\ 0.0002 \\ 0.03 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$c_1 = \frac{-7.887\times10^3 - (-9.2307\times10^5)c_2 - 0c_3 - 0c_4}{4.2857\times10^7}$$

$$c_2 = \frac{0 - 4.2857\times10^7 c_1 - (-4.2857\times10^7)c_3 - 5.4619\times10^5 c_4}{-5.4619\times10^5}$$

$$c_3 = \frac{0.007 - (-6.5)c_1 - (-0.15384)c_2 - 0.15384c_4}{6.5}$$

$$c_4 = \frac{0 - 0c_1 - 0c_2 - 4.2857\times 10^7 c_3}{-3.6057\times 10^5}$$

<u>Iteration #1</u>

Given the initial guess of the solution vector as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -0.005 \\ 0.001 \\ 0.0002 \\ 0.03 \end{bmatrix}$$

we get

$$c_1 = \frac{-7.887\times 10^3 - \left(-9.2307\times 10^5\right)\times 0.001}{4.2857\times 10^7}$$

$$= -1.6249\times 10^{-4}$$

$$c_2 = \frac{0 - 4.2857\times 10^7 \times \left(-1.6249\times 10^{-4}\right) - \left(-4.2857\times 10^7\right)\times 0.0002 - 5.4619\times 10^5 \times 0.03}{-5.4619\times 10^5}$$

$$= 1.5569\times 10^{-3}$$

$$c_3 = \frac{0.007 - \left(-6.5\right)\times\left(-1.6249\times 10^{-4}\right) - \left(-0.15384\right)\times 1.5569\times 10^{-3} - 0.15384\times 0.03}{6.5}$$

$$= 2.4125\times 10^{-4}$$

$$c_4 = \frac{0 - 4.2857\times 10^7 \times 2.4125\times 10^{-4}}{-3.6057\times 10^5}$$

$$= 2.8675\times 10^{-2}$$

The absolute relative approximate error for each c_i then is

$$\left|\in_a\right|_1 = \left|\frac{-1.6249\times 10^{-4} - \left(-0.005\right)}{-1.6249\times 10^{-4}}\right|\times 100$$

$$= 2977.1\%$$

$$\left|\in_a\right|_2 = \left|\frac{1.5569\times 10^{-3} - 0.001}{1.5569\times 10^{-3}}\right|\times 100$$

$$= 35.770\%$$

$$\left|\in_a\right|_3 = \left|\frac{2.4125\times 10^{-4} - 0.002}{2.4125\times 10^{-4}}\right|\times 100$$

$$= 17.098\%$$

$$\left|\in_a\right|_4 = \left|\frac{2.8675\times 10^{-2} - 0.03}{2.8675\times 10^{-2}}\right|\times 100$$

$$= 4.6223\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.6249\times10^{-4} \\ 1.5569\times10^{-3} \\ 2.4125\times10^{-4} \\ 2.8675\times10^{-2} \end{bmatrix}$$

and the maximum absolute relative approximate error is 2977.1%.

<u>Iteration #2</u>

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.6249\times10^{-4} \\ 1.5569\times10^{-3} \\ 2.4125\times10^{-4} \\ 2.8675\times10^{-2} \end{bmatrix}$$

Now we get

$$c_1 = \frac{-7.887\times10^3 - \left(-9.2307\times10^5\right)\times1.5569\times10^{-3}}{4.2857\times10^7}$$

$$= -1.5050\times10^{-4}$$

$$c_2 = \frac{\begin{pmatrix} 0 - 4.2857\times10^7 \times\left(-1.5050\times10^{-4}\right) - \left(-4.2857\times10^7\right)\times2.4125\times10^{-4} \\ -5.4619\times10^5\times2.8675\times10^{-2} \end{pmatrix}}{-5.4619\times10^5}$$

$$= -2.0639\times10^{-3}$$

$$c_3 = \frac{\begin{pmatrix} 0.007 - \left(-6.5\right)\times\left(-1.5050\times10^{-4}\right) - \left(-0.15384\right)\times -2.0639\times10^{-3} \\ -0.15384\times2.8675\times10^{-2} \end{pmatrix}}{6.5}$$

$$= 1.9892\times10^{-4}$$

$$c_4 = \frac{0 - 4.2857\times10^7\times1.9892\times10^{-4}}{-3.6057\times10^5}$$

$$= 2.3643\times10^{-2}$$

The absolute relative approximate error for each c_i then is

$$\left|\in_a\right|_1 = \left|\frac{-1.5050\times10^{-4} - \left(-1.6249\times10^{-4}\right)}{-1.5050\times10^{-4}}\right|\times100$$

$$= 7.9702\%$$

$$\left|\in_a\right|_2 = \left|\frac{-2.0639\times10^{-3} - 1.5569\times10^{-3}}{-2.0639\times10^{-3}}\right|\times100$$

$$= 175.44\%$$

$$\left|\in_a\right|_3 = \left|\frac{1.9892\times10^{-4} - 2.4125\times10^{-4}}{1.9892\times10^{-4}}\right|\times100$$

$$= 21.281\%$$

$$|\in_a|_4 = \left|\frac{2.3643\times10^{-2} - 2.8675\times10^{-2}}{2.3643\times10^{-2}}\right|\times100$$
$$= 21.281\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.5050\times10^{-4} \\ -2.0639\times10^{-3} \\ 1.9892\times10^{-4} \\ 2.3643\times10^{-2} \end{bmatrix}$$

and the maximum absolute relative approximate error is 175.44%.

At the end of the second iteration the stress on the inside radius of the outer cylinder is calculated

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right]$$
$$= \frac{30\times10^6}{1-(0.3)^2}\left[1.9892\times10^{-4}(1+0.3) + 2.3643\times10^{-2}\left(\frac{1-0.3}{(6.5)^2}\right)\right]$$
$$= 21439 \text{ psi}$$

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	c_1	$\|\in_a\|_1 \%$	c_2	$\|\in_a\|_2 \%$
1	-1.6249×10^{-4}	2977.1	1.5569×10^{-3}	35.770
2	-1.5050×10^{-4}	7.9702	-2.0639×10^{-3}	175.44
3	-2.2848×10^{-4}	34.132	-9.8931×10^{-3}	79.138
4	-3.9711×10^{-4}	42.464	-2.8949×10^{-2}	65.826
5	-8.0755×10^{-4}	50.825	-6.9799×10^{-2}	58.524
6	-1.6874×10^{-3}	52.142	-1.7015×10^{-1}	58.978

Iteration	c_3	$\|\in_a\|_3 \%$	c_4	$\|\in_a\|_4 \%$
1	2.4125×10^{-4}	17.098	2.8675×10^{-2}	4.6223
2	1.9892×10^{-4}	21.281	2.3643×10^{-2}	21.281
3	5.4716×10^{-5}	263.55	6.5035×10^{-3}	263.55
4	-1.5927×10^{-4}	134.35	-1.8931×10^{-2}	134.35
5	-9.3454×10^{-4}	82.957	-1.1108×10^{-1}	82.957
6	-2.0085×10^{-3}	53.472	-2.3873×10^{-1}	53.472

After six iterations, the absolute relative approximate errors are not decreasing. In fact, conducting more iterations reveals that the absolute relative approximate error does not approach zero or converge to any other number.

SIMULTANEOUS LINEAR EQUATIONS	
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