Chapter 04.07
LU Decomposition – More Examples
Civil Engineering

Example 1

To find the maximum stresses in a compound cylinder, the following four simultaneous linear equations need to solved.

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 4.2857\times10^7 & -5.4619\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

In the compound cylinder, the inner cylinder has an internal radius of $a = 5"$, and an outer radius $c = 6.5"$, while the outer cylinder has an internal radius of $c = 6.5"$ and an outer radius of $b = 8"$. Given $E = 30\times10^6$ psi, $\nu = 0.3$, and that the hoop stress in the outer cylinder is given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right],$$

find the stress on the inside radius of the outer cylinder.

Find the values of c_1, c_2, c_3 and c_4 using LU decomposition.

Solution

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

The $[U]$ matrix is the same as the one found at the end of the forward elimination steps of the naïve Gauss elimination method.

Forward Elimination of Unknowns

Since there are four equations, there will be three steps of forward elimination of unknowns.

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 4.2857\times10^7 & -5.4619\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}$$

First step

Divide Row 1 by 4.2857×10^{7} and multiply it by 4.2857×10^{7}, that is, multiply Row 1 by $4.2857\times10^{7}/4.2857\times10^{7}=1$. Then subtract the result from Row 2.

$$\text{Row }2-(\text{Row }1\times(1))=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ -6.5 & -0.15384 & 6.5 & 0.15384\\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5}\end{bmatrix}$$

Divide Row 1 by 4.2857×10^{7} and multiply it by -6.5, that is, multiply Row 1 by $-6.5/4.2857\times10^{7}=-1.5167\times10^{-7}$. Then subtract the result from Row 3.

$$\text{Row }3-(\text{Row }1\times(-1.5167\times10^{-7}))=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & -0.29384 & 6.5 & 0.15384\\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5}\end{bmatrix}$$

Divide Row 1 by 4.2857×10^{7} and multiply it by 0, that is, multiply Row 1 by $0/4.2857\times10^{7}=0$. Then subtract the result from Row 4.

$$\text{Row }4-(\text{Row }1\times(0))=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & -0.29384 & 6.5 & 0.15384\\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5}\end{bmatrix}$$

Second step

Divide Row 2 by 3.7688×10^{5} and multiply it by -0.29384, that is, multiply Row 2 by $-0.29384/3.7688\times10^{5}=-7.7966\times10^{-.7}$. Then subtract the result from Row 3.

$$\text{Row }3-(\text{Row }2\times(-7.7966\times10^{-.7}))=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & 0 & -26.914 & 0.57968\\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5}\end{bmatrix}$$

Divide Row 2 by 3.7688×10^{5} and multiply it by 0 that is, multiply Row 2 by $0/3.7688\times10^{5}=0$. Then subtract the result from Row 4.

$$\text{Row }4-(\text{Row }2\times(0))=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & 0 & -26.914 & 0.57968\\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5}\end{bmatrix}$$

Third step

Divide Row 3 by -26.914 and multiply it by 4.2857×10^{7} that is, multiply Row 3 by $4.2857\times10^{7}/-26.914=-1.5924\times10^{6}$. Then subtract the result from Row 4.

$$\text{Row 4}-\left(\text{Row 3}\times\left(-1.5924\times10^{6}\right)\right)=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & 0 & -26.914 & 0.57968\\ 0 & 0 & 0 & 5.6250\times10^{5}\end{bmatrix}$$

The coefficient matrix after the completion of the forward elimination steps is

$$[U]=\begin{bmatrix}4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0\\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5}\\ 0 & 0 & -26.9140 & 0.579684\\ 0 & 0 & 0 & 5.62500\times10^{5}\end{bmatrix}$$

Now find $[L]$.

$$[L]=\begin{bmatrix}1 & 0 & 0 & 0\\ \ell_{21} & 1 & 0 & 0\\ \ell_{31} & \ell_{32} & 1 & 0\\ \ell_{41} & \ell_{42} & \ell_{43} & 1\end{bmatrix}$$

From the first step of forward elimination,

$$\ell_{21}=\frac{4.2857\times10^{7}}{4.2857\times10^{7}}=1$$

$$\ell_{31}=\frac{-6.5}{4.2857\times10^{7}}=-1.5167\times10^{-7}$$

$$\ell_{41}=\frac{0}{4.2857\times10^{7}}=0$$

From the second step of forward elimination,

$$\ell_{32}=\frac{-0.29384}{3.7688\times10^{5}}=-7.7966\times10^{-7}$$

$$\ell_{42}=\frac{0}{3.7688\times10^{5}}=0$$

From the third step of forward elimination,

$$\ell_{43}=\frac{4.2857\times10^{7}}{-26.914}=-1.5294\times10^{6}$$

Hence

$$[L]=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ -1.5167\times10^{-7} & -7.7966\times10^{-7} & 1 & 0\\ 0 & 0 & -1.5924\times10^{6} & 1\end{bmatrix}$$

Now that $[L]$ and $[U]$ are known, solve $[L][Z]=[C]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1.5167\times10^{-7} & -7.7966\times10^{-7} & 1 & 0 \\ 0 & 0 & -1.5924\times10^{6} & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

to give

$$z_1 = -7.887\times10^{3}$$
$$z_1 + z_2 = 0$$
$$-1.5167\times10^{-7} z_1 + \left(-7.7966\times10^{-7}\right)z_2 + z_3 = 0.007$$
$$-1.5924\times10^{6} z_3 + z_4 = 0$$

Forward substitution starting from the first equation gives

$$z_1 = -7.887\times10^{3}$$
$$\begin{aligned} z_2 &= -z_1 \\ &= -\left(-7.887\times10^{3}\right) \\ &= 7.887\times10^{3} \end{aligned}$$
$$\begin{aligned} z_3 &= 0.007 - \left(-1.5167\times10^{-7}\right)z_1 - \left(-7.7966\times10^{-7}\right)z_2 \\ &= 0.007 - \left(-1.51667\times10^{-7}\right)\times\left(-7.887\times10^{3}\right) - \left(-7.79662\times10^{-7}\right)\times\left(7.887\times10^{3}\right) \\ &= 1.1953\times10^{-2} \end{aligned}$$
$$\begin{aligned} z_4 &= -\left(-1.5924\times10^{6}\right)z_3 \\ &= -\left(-1.5924\times10^{6}\right)\times\left(1.1953\times10^{-2}\right) \\ &= 19034 \end{aligned}$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 19034 \end{bmatrix}$$

Now solve

$$[U][C] = [Z]$$
$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.6250\times10^{5} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 19034 \end{bmatrix}$$

$$4.2857\times10^{7} c_1 + \left(-9.2307\times10^{5}\right)c_2 + (0)c_3 + (0)c_4 = -7.887\times10^{3}$$
$$3.7688\times10^{5} c_2 + \left(-4.2857\times10^{7}\right)c_3 + 5.4619\times10^{5} c_4 = 7.887\times10^{3}$$
$$-26.914 c_3 + 0.57968 c_4 = 1.1953\times10^{-2}$$
$$5.6250\times10^{5} c_4 = 19034$$

From the fourth equation,

$$5.6250\times10^{5}c_4 = 19034$$

$$c_4 = \frac{19034}{5.6250\times10^{5}}$$

$$= 3.3837\times10^{-2}$$

Substituting the value of c_4 into the third equation,

$$-26.914c_3 + 0.57968c_4 = 1.1953\times10^{-2}$$

$$c_3 = \frac{1.1953\times10^{-2} - 0.57968c_4}{-26.9140}$$

$$= \frac{1.1953\times10^{-2} - 0.57968\times(3.3837\times10^{-2})}{-26.9140}$$

$$= 2.8469\times10^{-4}$$

Substituting the values of c_3 and c_4 into the second equation,

$$3.7688\times10^{5}c_2 + \left(-4.2857\times10^{7}\right)c_3 + 5.4619\times10^{5}c_4 = 7.887\times10^{3}$$

$$c_2 = \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)c_3 - 5.4619\times10^{5}c_4}{3.7668\times10^{5}}$$

$$= \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)\times\left(2.84687\times10^{-4}\right) - 5.4619\times10^{5}\times\left(3.3838\times10^{-2}\right)}{3.7688\times10^{5}}$$

$$= 4.2615\times10^{-3}$$

Substituting the values of c_2, c_3 and c_4 into the first equation,

$$4.2857\times10^{7}c_1 + \left(-9.2307\times10^{5}\right)c_2 + (0)c_3 + (0)c_4 = -7.887\times10^{3}$$

$$c_1 = \frac{-7.887\times10^{3} - \left(-9.2307\times10^{5}\right)c_2}{4.2857\times10^{7}}$$

$$= \frac{-7.887\times10^{3} - \left(-9.2307\times10^{5}\right)\times\left(4.2615\times10^{-3}\right)}{4.2857\times10^{7}}$$

$$= 9.2244\times10^{-5}$$

The solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -9.2244\times10^{-5} \\ 4.2615\times10^{-3} \\ 2.8469\times10^{-4} \\ 3.3837\times10^{-2} \end{bmatrix}$$

The stress on the inside radius of the outer cylinder is then given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right]$$

$$= \frac{30\times10^{6}}{1-0.3^2}\left[2.8469\times10^{-4}(1+0.3) + 3.3837\times10^{-2}\left(\frac{1-0.3}{6.5^2}\right)\right]$$

$$= 30683\ \text{psi}$$

SIMULTANEOUS LINEAR EQUATIONS	
Topic	LU Decomposition – More Examples
Summary	Examples of LU decomposition
Major	Civil Engineering
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