Chapter 04.06
Gaussian Elimination – More Examples
Civil Engineering

Example 1

To find the maximum stresses in a compound cylinder, the following four simultaneous linear equations need to be solved.

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

In the compound cylinder, the inner cylinder has an internal radius of $a = 5"$ and an outer radius of $c = 6.5"$, while the outer cylinder has an internal radius of $c = 6.5"$ and an outer radius of $b = 8"$. Given $E = 30\times10^{6}\text{ psi}$, $\nu = 0.3$, and that the hoop stress in the outer cylinder is given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right],$$

find the stress on the inside radius of the outer cylinder.
Find the values of c_1, c_2, c_3 and c_4 using naïve Gauss elimination.

Solution

Forward Elimination of Unknowns

Since there are four equations, there will be three steps of forward elimination of unknowns.

<u>First step</u>
Divide Row 1 by 4.2857×10^{7} and then multiply it by 4.2857×10^{7}, that is, multiply Row 1 by $4.2857\times10^{7}/4.2857\times10^{7} = 1$.

$$\text{Row }1\times(1) = \begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -7.887\times10^{3} \end{bmatrix}$$

Subtract the result from Row 2 to get

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 0 & 3.7688\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 7.887\times10^3 \\ 0.007 \\ 0 \end{bmatrix}$$

Divide Row 1 by 4.2857×10^7 and then multiply it by -6.5, that is, multiply Row 1 by $-6.5/4.2857\times10^7 = -1.5167\times10^{-7}$.

$$\text{Row }1\times\left(-1.5167\times10^{-7}\right) = \begin{bmatrix} -6.5 & 0.14000 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1.1962\times10^{-3} \end{bmatrix}$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 0 & 3.7688\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 7.887\times10^3 \\ 5.8038\times10^{-3} \\ 0 \end{bmatrix}$$

Divide Row 1 by 4.2857×10^7 and then multiply it by 0, that is, multiply Row 1 by $0/4.2857\times10^7 = 0$.

$$\text{Row }1\times(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

Subtract the result from Row 4 to get

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 0 & 3.7688\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 7.887\times10^3 \\ 5.8038\times10^{-3} \\ 0 \end{bmatrix}$$

<u>Second step</u>

We now divide Row 2 by 3.7688×10^5 and then multiply it by -0.29384, that is, multiply Row 2 by $-0.29384/3.7688\times10^5 = -7.7966\times10^{-7}$.

$$\text{Row }2\times\left(-7.7966\times10^{-7}\right) =$$

$$\begin{bmatrix} 0 & -0.29384 & 33.414 & -0.42584 \end{bmatrix} \quad \begin{bmatrix} -6.1492\times10^{-3} \end{bmatrix}$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 0 & 3.7688\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^3 \\ 7.887\times10^3 \\ 1.1953\times10^{-2} \\ 0 \end{bmatrix}$$

Divide Row 2 by 3.7688×10^5 and then multiply by 0, that is, multiply Row 2 by $0/3.7688\times10^5 = 0$.

$$\text{Row }2\times(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

Subtract the result from Row 4 to get

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 0 \end{bmatrix}$$

Third step

We now divide Row 3 by -26.914 and then multiply by 4.2857×10^{7}, that is, multiply Row 2 by $4.2857\times10^{7}/-26.914=-1.5924\times10^{6}$.

$$\text{Row }3\times\left(-1.5924\times10^{6}\right)= \begin{bmatrix}0 & 0 & 4.2857\times10^{7} & -9.2307\times10^{5}\end{bmatrix} \qquad \begin{bmatrix}-1.9034\times10^{4}\end{bmatrix}$$

Subtract the result from Row 4 to get

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.625\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 1.9034\times10^{4} \end{bmatrix}$$

Back substitution

From the fourth equation,

$$5.625\times10^{5}c_4 = 1.9034\times10^{4}$$

$$c_4 = \frac{1.9034\times10^{4}}{5.625\times10^{5}}$$

$$= 3.3837\times10^{-2}$$

Substituting the value of c_4 in the third equation,

$$-26.914c_3 + (0.57968)c_4 = 1.1953\times10^{-2}$$

$$c_3 = \frac{1.1953\times10^{-2} - (0.57968)c_4}{-26.914}$$

$$= \frac{1.1953\times10^{-2} - (0.57968)\times3.3837\times10^{-2}}{-26.914}$$

$$= 2.8469\times10^{-4}$$

Substituting the value of c_3 and c_4 in the second equation,

$$3.7688\times10^{5}c_2 + \left(-4.2857\times10^{7}\right)c_3 + 5.4619\times10^{5}c_4 = 7.887\times10^{3}$$

$$c_2 = \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)c_3 - 5.4619\times10^{5}c_4}{3.7688\times10^{5}}$$

$$= \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)\times\left(2.8469\times10^{-4}\right) - \left(5.4619\times10^{5}\right)\times\left(3.3837\times10^{-2}\right)}{3.7688\times10^{5}}$$

$$= 4.2615\times10^{-3}$$

Substituting the values of c_2, c_3 and c_4 in the first equation,

$$4.2857\times10^{7}c_1+\left(-9.2307\times10^{5}\right)c_2+(0)c_3+(0)c_4=-7.887\times10^{3}$$

$$c_1=\frac{-7.887\times10^{3}-\left(-9.2307\times10^{5}\right)c_2-(0)c_3-(0)c_4}{4.2857\times10^{7}}$$

$$=\frac{-7.887\times10^{3}-\left(-9.2307\times10^{5}\right)\times\left(4.2615\times10^{-3}\right)}{4.2857\times10^{7}}$$

$$=-9.2244\times10^{-5}$$

Hence the solution vector is

$$\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}-9.2244\times10^{-5}\\4.2615\times10^{-3}\\2.8469\times10^{-4}\\3.3837\times10^{-2}\end{bmatrix}$$

The stress on the inside radius of the outer cylinder is then given by

$$\sigma_\theta=\frac{E}{1-\nu^2}\left[c_3(1+\nu)+c_4\left(\frac{1-\nu}{r^2}\right)\right]$$

$$=\frac{30\times10^{6}}{1-0.3^2}\left[2.8469\times10^{-4}(1+0.3)+3.3837\times10^{-2}\left(\frac{1-0.3}{6.5^2}\right)\right]$$

$$=30683\,\text{psi}$$

SIMULTANEOUS LINEAR EQUATIONS	
Topic	Gaussian Elimination – More Examples
Summary	Examples of Gaussian elimination
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Authors	Autar Kaw
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