Chapter 02.03
Differentiation of Discrete Functions-More Examples
Civil Engineering

Example 1

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 1 the radial displacements u are given along the y-axis. The radius of the hole is 1.0 cm.

a) At $x = 0$, if the radial strain ε_r is given by $\varepsilon_r = \dfrac{\partial u}{\partial r}$, find the radial strain at $r = 1.1\text{cm}$ using the forward divided difference method.

b) If the tangential strain at $r = 1.1\text{cm}$, $\theta = 90°$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ, at $r = 1.1\,\text{cm}$, $\theta = 90°$ if $\sigma_\theta = \dfrac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200\,\text{GPa}$ and $\nu = 0.3$.

Table 1 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	–0.0010000
1.1	–0.0010689
1.2	–0.0011088
1.3	–0.0011326
1.4	–0.0011474
1.5	–0.0011574
1.6	–0.0011650
1.7	–0.0011718
1.8	–0.0011785
1.9	–0.0011857

Solution

a)

$$\varepsilon_r = \frac{\partial u}{\partial r}$$
$$\approx \frac{u_{i+1} - u_i}{\Delta r}$$
$$r_i = 1.1$$
$$r_{i+1} = 1.2$$
$$\Delta r = r_{i+1} - r_i$$
$$= 1.2 - 1.1$$
$$= 0.1$$
$$u_{i+1} = -0.0011088$$
$$u_i = -0.0010689$$
$$\varepsilon_r = \frac{u_{i+1} - u_i}{\Delta r}$$
$$= \frac{-0.0011088 - (-0.0010689)}{0.1}$$
$$= -0.000399 \text{ cm/cm}$$

b) $$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$
$$= \frac{2\times 10^{11}}{1-0.3^2}(-0.000399 + 0.3\times 0.0029733)$$
$$= 108.35\times 10^6 \text{ Pa}$$

Example 2

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 2 the radial displacements u are given along the y-axis. The radius of the hole is 1.0 cm.

a) At $x = 0$, if the radial strain ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1 \text{ cm}$. Use a third order polynomial interpolant for calculating the radial strain.

b) If the tangential strain at $r = 1.1\text{cm}$, $\theta = 90°$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ, at $r = 1.1\text{cm}$, $\theta = 90°$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200\,\text{GPa}$ and $\nu = 0.3$.

Table 2 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	–0.0010000
1.1	–0.0010689
1.2	–0.0011088
1.3	–0.0011326
1.4	–0.0011474
1.5	–0.0011574
1.6	–0.0011650
1.7	–0.0011718
1.8	–0.0011785
1.9	–0.0011857

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the displacement given by

$$u(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$

Since we want to find the radial strain at $r = 1.1$ cm, and we are using a third order polynomial, we need to choose the four points closest to $r = 1.1$ that also bracket $r = 1.1$ to evaluate it.

The four points are $r_0 = 1.0$, $r_1 = 1.1$, $r_2 = 1.2$ and $r_3 = 1.3$.

$$r_0 = 1.0,\ u(r_0) = -0.0010000$$

$$r_1 = 1.1,\ u(r_1) = -0.0010689$$

$$r_2 = 1.2,\ u(r_2) = -0.0011088$$

$$r_3 = 1.3,\ u(r_3) = -0.0011326$$

such that

$$u(1.0) = -0.0010000 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$

$$u(1.1) = -0.0010689 = a_0 + a_1(1.1) + a_2(1.1)^2 + a_3(1.1)^3$$

$$u(1.2) = -0.0011088 = a_0 + a_1(1.2) + a_2(1.2)^2 + a_3(1.2)^3$$

$$u(1.3) = -0.0011326 = a_0 + a_1(1.3) + a_2(1.3)^2 + a_3(1.3)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.1 & 1.21 & 1.331 \\ 1 & 1.2 & 1.44 & 1.728 \\ 1 & 1.3 & 1.69 & 2.197 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.0010000 \\ -0.0010689 \\ -0.0011088 \\ -0.0011326 \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.004122$$

$$a_1 = -0.011517$$

$$a_2 = 0.008545$$

$$a_3 = -0.00215$$

Hence

$$u(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$
$$= 0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3, \quad 1 \le r \le 1.3$$

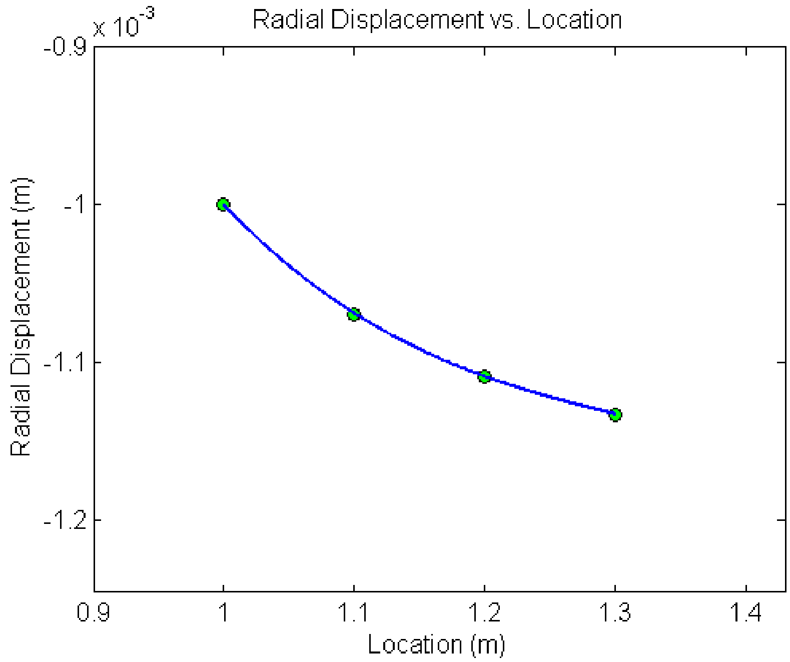

Figure 1 Graph of radial displacement vs. location.

The derivative of radial displacement at $r = 1.1\text{cm}$ is given by

$$u'(1.1) = \frac{d}{dr} u(r)\Big|_{r=1.1}$$

Given that $u(r) = 0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3, \ 1 \le r \le 1.3$,

$$u'(r) = \frac{d}{dr} u(r)$$
$$= \frac{d}{dr}\left(0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3\right)$$
$$= -0.011517 + 0.01709r - 0.00645r^2, \quad 1 \le r \le 1.3$$
$$u'(1.1) = -0.011517 + 0.01709(1.1) - 0.00645(1.1)^2$$
$$= -0.0005225 \text{ cm/cm}$$

$$\varepsilon_r = -0.00052250 \text{ cm/cm}$$

b) $$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$= \frac{2\times10^{11}}{1-0.3^2}(-0.00052250 + 0.3\times0.0029733)$$
$$= 81.207\times10^{6} \text{ Pa}$$

Example 3

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 3 the radial displacements u are given along the y-axis. The radius of the hole is 1.0 cm.

a) At , $x = 0$ if the radial strain ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1$ cm. Use a second order Lagrange polynomial interpolant for calculating the radial strain.
b) If the tangential strain at $r = 1.1$ cm, $\theta = 90°$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ, at $r = 1.1$ cm, $\theta = 90°$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200$ GPa and $\nu = 0.3$.

Table 3 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	–0.0010000
1.1	–0.0010689
1.2	–0.0011088
1.3	–0.0011326
1.4	–0.0011474
1.5	–0.0011574
1.6	–0.0011650
1.7	–0.0011718
1.8	–0.0011785
1.9	–0.0011857

Solution

For second order Lagrangian interpolation, we choose the radial displacement given by

$$u(r) = \left(\frac{r-r_1}{r_0-r_1}\right)\left(\frac{r-r_2}{r_0-r_2}\right)u(r_0) + \left(\frac{r-r_0}{r_1-r_0}\right)\left(\frac{r-r_2}{r_1-r_2}\right)u(r_1) + \left(\frac{r-r_0}{r_2-r_0}\right)\left(\frac{r-r_1}{r_2-r_1}\right)u(r_2)$$

(a) Change in the radial displacement at 1.1 cm:

Since we want to find the rate of change in the radial displacement at $r = 1.1\,\text{cm}$, and we are using second order Lagrangian interpolation, we need to choose the three points closest to $r = 1.1\,\text{cm}$ that also bracket $r = 1.1\,\text{cm}$ to evaluate it.

The three points are $r_0 = 1.0$, $r_1 = 1.1$, and $r_2 = 1.2$.

$$r_0 = 1.0,\ u(r_0) = -0.0010000$$
$$r_1 = 1.1,\ u(r_1) = -0.0010689$$
$$r_2 = 1.2,\ u(r_2) = -0.0011088$$

The change in the radial displacement at $r = 1.1\,\text{cm}$ is given by

$$\frac{du(1.1)}{dr} = \frac{d}{dr}u(r)\Big|_{r=1.1}$$

Hence

$$u'(r) = \frac{2r-(r_1+r_2)}{(r_0-r_1)(r_0-r_2)}u(r_0) + \frac{2r-(r_0+r_2)}{(r_1-r_0)(r_1-r_2)}u(r_1) + \frac{2r-(r_0+r_1)}{(r_2-r_0)(r_2-r_1)}u(r_2)$$

$$u'(1.1) = \frac{2(1.1)-(1.1+1.2)}{(1.0-1.1)(1.0-1.2)}(-0.0010000) + \frac{2(1.1)-(1.0+1.2)}{(1.1-1.0)(1.1-1.2)}(-0.0010689) + \frac{2(1.1)-(1.0+1.1)}{(1.2-1.0)(1.2-1.1)}(-0.0011088)$$

$$= -5(-0.0010000) + 0(-0.0010689) + 5(-0.0011088)$$
$$= -0.000544\ \text{cm/cm}$$

b)

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$
$$= \frac{2\times10^{11}}{1-0.3^2}(-0.000544 + 0.3\times0.0029733)$$
$$= 76.481\times10^6\ \text{Pa}$$

DIFFERENTIATION	
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Summary	Examples of Discrete Functions
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