Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Chemical Engineering

Example 1
The concentration of salt $x$ in a home made soap maker is given as a function of time by
\[
\frac{dx}{dt} = 37.5 - 3.5x
\]
At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L. Using Runge-Kutta 2nd order method and a step size of $h = 1.5$ min, what is the salt concentration after 3 minutes?

Solution
\[
\frac{dx}{dt} = 37.5 - 3.5x
\]
\[
f(t,x) = 37.5 - 3.5x
\]
Per Heun's method
\[
x_{i+1} = x_i + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2\right) h
\]
\[
k_1 = f(t_i, x_i)
\]
\[
k_2 = f(t_i + h, x_i + k_1 h)
\]
For $i = 0$, $t_0 = 0$, $x_0 = 50$
\[
k_1 = f(t_0, x_0)
\]
\[
= f(0,50)
\]
\[
= 37.5 - 3.5(50)
\]
\[
= -137.5
\]
\[
k_2 = f(t_0 + h, x_0 + k_1 h)
\]
\[
= f(0 + 1.5, 50 + (-137.5)1.5)
\]
\[
= f(1.5, -156.25)
\]
\[
= 37.5 - 3.5(-156.25)
\]
\[
= 584.38
\]
\[
x_1 = x_0 + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2\right) h
\]
\[
\begin{align*}
&= 50 + \left( \frac{1}{2}(-137.5) + \frac{1}{2}(584.38) \right) \times 1.5 \\
&= 50 + (223.44) \times 1.5 \\
&= 385.16 \text{ g/L}
\end{align*}
\]

\(x_1\) is the approximate concentration of salt at

\(t = t_1 = t_0 + h = 0 + 1.5 = 1.5 \text{ min} \)
\(x(1.5) \approx x_1 = 385.16 \text{ g/L} \)

For \(i = 1\), \(t_i = t_0 + h = 0 + 1.5 = 1.5\), \(x_1 = 385.16\)

\[
k_1 = f(t_1, x_1)\\
= f(1.5, 385.16)\\
= 37.5 - 3.5(385.16)\\
= -1310.5
\]

\[
k_2 = f(t_1 + h, x_1 + k_1 h)\\
= f(1.5 + 1.5, 385.16 + (-1310.5) \times 1.5)\\
= f(3, -1580.6)\\
= 37.5 - 3.5(-1580.6)\\
= 5569.8
\]

\[
x_2 = x_1 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h
\]
\[
= 385.16 + \left( \frac{1}{2}(-1310.5) + \frac{1}{2}(5569.8) \right) \times 1.5
\]
\[
= 385.16 + (2129.6) \times 1.5
\]
\[
= 3579.6 \text{ g/L}
\]

\(x(3) \approx x_2 = 3579.7 \text{ g/L} \)

\(x_2\) is the approximate concentration of salt at

\(t = t_2 = t_1 + h = 1.5 + 1.5 = 3 \text{ min} \)
\(x(3) \approx x_2 = 3579.7 \text{ g/L} \)

The results from Heun’s method are compared with exact results in Figure 1.

The exact solution of the ordinary differential equation is given by

\[
x(t) = 10.714 + 39.286e^{-3.5t}
\]

The solution to this nonlinear equation at \(t = 3\text{ min}\) is

\[
x(3) = 10.715 \text{ g/L}
\]
Using smaller step size would increase the accuracy of the result as given in Table 1 and Figure 2.

**Table 1** Effect of step size for Heun’s method.

| Step Size, $h$ | $x(3)$  | $E_r$   | $|\epsilon_r|$ | %  |
|---------------|---------|---------|------------------|----|
| 3             | 1803.1  | -1792.4 | 16727            |    |
| 1.5           | 3579.6  | -3568.9 | 33306            |    |
| 0.75          | 442.05  | -431.34 | 4025.4           |    |
| 0.375         | 11.038  | -0.32231| 3.0079           |    |
| 0.1875        | 10.718  | -0.0024979 | 0.023311   |    |
In Table 2, the Euler’s method and Runge-Kutta 2nd order method results are shown as a function of step size.

**Table 2** Comparison of Euler and the Runge-Kutta methods.

<table>
<thead>
<tr>
<th>Step size, $h$</th>
<th>$x(3)$</th>
<th>Euler</th>
<th>Heun</th>
<th>Midpoint</th>
<th>Ralston</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$-362.50$</td>
<td>1803.1</td>
<td>1803.1</td>
<td>1803.1</td>
<td>1803.1</td>
</tr>
<tr>
<td>1.5</td>
<td>720.31</td>
<td>3579.6</td>
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<td>284.65</td>
<td>442.05</td>
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<tr>
<td>0.375</td>
<td>10.718</td>
<td>11.038</td>
<td>11.038</td>
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</tr>
<tr>
<td>0.1875</td>
<td>10.714</td>
<td>10.718</td>
<td>10.718</td>
<td>10.718</td>
<td>10.718</td>
</tr>
</tbody>
</table>

Figure 3 shows the comparison of Euler and Runge-Kutta methods over time.
Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.