Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Chemical Engineering

Example 1

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L. Using Runge-Kutta 2nd order method and a step size of $h = 1.5 \text{ min}$, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t,x) = 37.5 - 3.5x$$

Per Heun's method

$$x_{i+1} = x_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(t_i, x_i)$$

$$k_2 = f(t_i + h, x_i + k_1 h)$$

For $i = 0,\ t_0 = 0,\ x_0 = 50$

$$\begin{aligned} k_1 &= f(t_0, x_0) \\ &= f(0,50) \\ &= 37.5 - 3.5(50) \\ &= -137.5 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_0 + h, x_0 + k_1 h) \\ &= f(0 + 1.5, 50 + (-137.5)1.5) \\ &= f(1.5, -156.25) \\ &= 37.5 - 3.5(-156.25) \\ &= 584.38 \end{aligned}$$

$$x_1 = x_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 50 + \left(\frac{1}{2}(-137.5) + \frac{1}{2}(584.38) \right)1.5$$
$$= 50 + (223.44)1.5$$
$$= 385.16 \text{ g/L}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5 \text{ min}$$
$$x(1.5) \approx x_1 = 385.16 \text{ g/L}$$

For $i = 1$, $t_1 = t_0 + h = 0 + 1.5 = 1.5$, $x_1 = 385.16$

$$k_1 = f(t_1, x_1)$$
$$= f(1.5, 385.16)$$
$$= 37.5 - 3.5(385.16)$$
$$= -1310.5$$
$$k_2 = f(t_1 + h, x_1 + k_1 h)$$
$$= f(1.5 + 1.5, 385.16 + (-1310.5)1.5)$$
$$= f(3, -1580.6)$$
$$= 37.5 - 3.5(-1580.6)$$
$$= 5569.8$$
$$x_2 = x_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2 \right)h$$
$$= 385.16 + \left(\frac{1}{2}(-1310.5) + \frac{1}{2}(5569.8) \right)1.5$$
$$= 385.16 + (2129.6)1.5$$
$$= 3579.6 \text{ g/L}$$
$$x(3) \approx x_2 = 3579.7 \text{ g/L}$$

x_2 is the approximate concentration of salt at

$$t = t_2 = t_1 + h = 1.5 + 1.5 = 3 \text{ min}$$
$$x(3) \approx x_2 = 3579.7 \text{ g/L}$$

The results from Heun's method are compared with exact results in Figure 1.

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3\,\text{min}$ is

$$x(3) = 10.715 \text{ g/L}$$

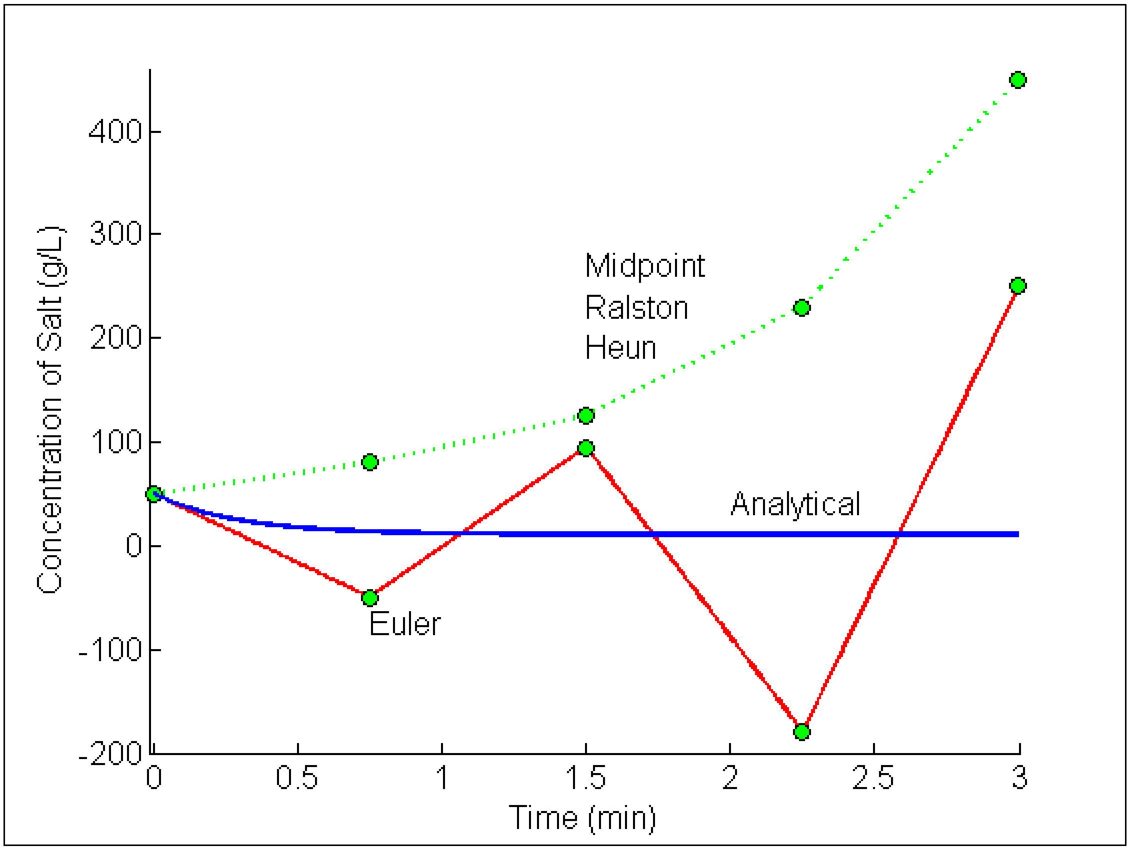

Figure 1 Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2.

Table 1 Effect of step size for Heun's method.

Step Size, h	$x(3)$	E_t	$\lvert\in_t\rvert$ %
3	1803.1	−1792.4	16727
1.5	3579.6	−3568.9	33306
0.75	442.05	−431.34	4025.4
0.375	11.038	−0.32231	3.0079
0.1875	10.718	−0.0024979	0.023311

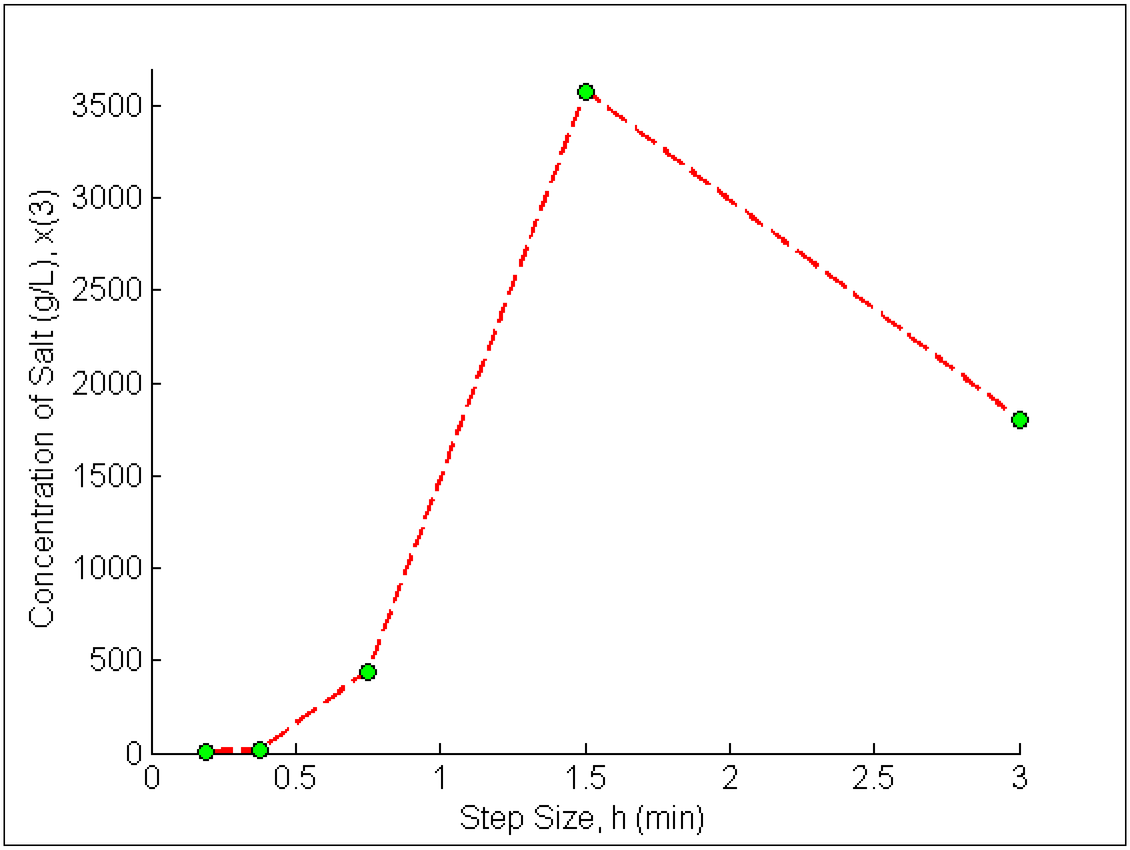

Figure 2 Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2nd order method results are shown as a function of step size.

Table 2 Comparison of Euler and the Runge-Kutta methods.

Step size, h	$x(3)$			
	Euler	Heun	Midpoint	Ralston
3	−362.50	1803.1	1803.1	1803.1
1.5	720.31	3579.6	3579.6	3579.6
0.75	284.65	442.05	442.05	442.05
0.375	10.718	11.038	11.038	11.038
0.1875	10.714	10.718	10.718	10.718

Figure 3 shows the comparison of Euler and Runge-Kutta methods over time.

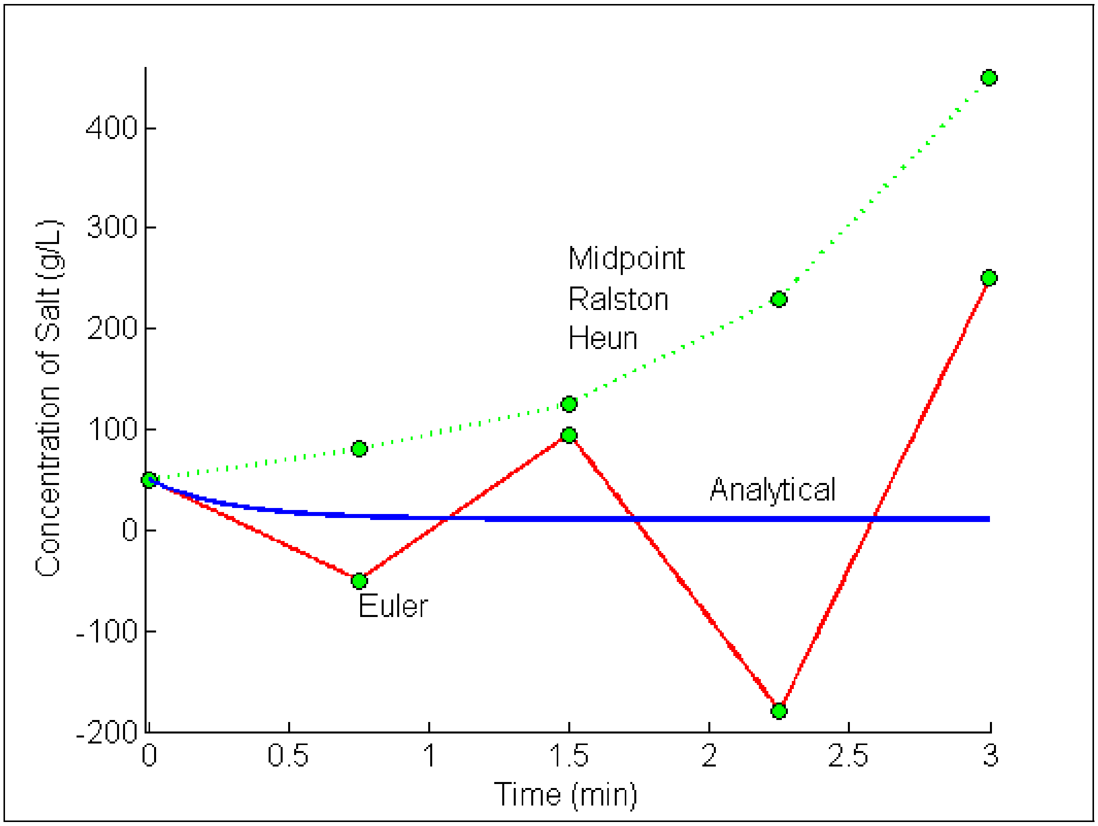

Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.