Chapter 05.05
Spline Method of Interpolation – More Examples
Chemical Engineering

Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at $61°\text{C}$. The specific heat of water is given as a function of time in Table 1.

Table 1 Specific heat of water as a function of temperature.

Temperature, T $(°\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg}-°\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

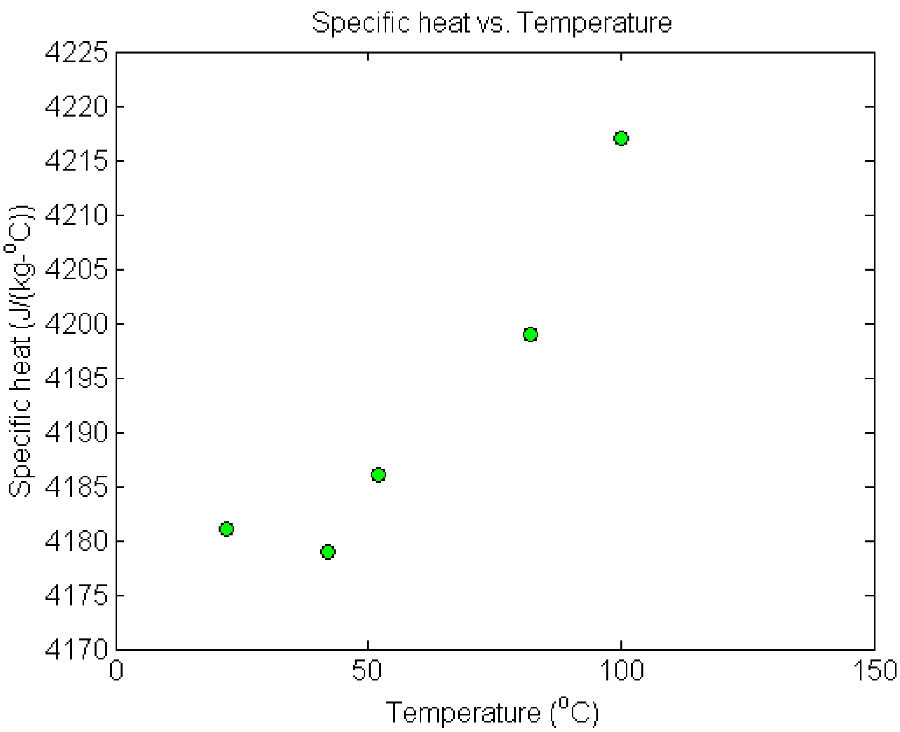

Figure 1 Specific heat of water vs. temperature.

Determine the value of the specific heat at $T = 61°\text{C}$ using linear splines.

Solution

Since we want to find the specific heat at $T = 61°\text{C}$ and we are using linear splines, we need to choose the two data points closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$ to evaluate it. The two points are $T_0 = 52$ and $T_1 = 82$.

Then

$$T_0 = 52, \quad C_p(T_0) = 4186$$

$$T_1 = 82, \quad C_p(T_1) = 4199$$

gives

$$C_p(T) = C_p(T_0) + \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}(T - T_0)$$

$$= 4186 + \frac{4199 - 4186}{82 - 52}(T - 52)$$

Hence

$$C_p(T) = 4186 + 0.43333(T - 52), \ 52 \le T \le 82$$

At $T = 61$,

$$C_p(61) = 4186 + 0.43333(61 - 52)$$

$$= 4189.9 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C. The specific heat of water is given as a function of time in Table 2.

Table 2 Specific heat of water as a function of temperature.

Temperature, T $(^\circ\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - {}^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

a) Determine the value of the specific heat at $T = 61^\circ\text{C}$ using quadratic splines. Find the absolute relative approximate error for the quadratic approximation.

b) The heat required to heat the water is given more accurately by

$$Q = m\int_{T_r}^{T_b} C_p dT$$

$T_r =$ room temperature $(^\circ\text{C})$

$T_b =$ boiling temperature of water $(^\circ\text{C})$

Given

$$T_r = 22^\circ\text{C}$$

$$T_b = 100^\circ\text{C}$$

find a better estimate of the heat required. What is the difference between the results from part (a) and part (b).

Solution

a) Since there are five data points, four quadratic splines pass through them.

$$\begin{aligned} C_p(T) &= a_1T^2 + b_1T + c_1, && 22 \le T \le 42 \\ &= a_2T^2 + b_2T + c_2, && 42 \le T \le 52 \\ &= a_3T^2 + b_3T + c_3, && 52 \le T \le 82 \end{aligned}$$

$$= a_4T^2 + b_4T + c_4, \qquad 82 \le T \le 100$$

The equations are found as follows

1. Each quadratic spline passes through two consecutive data points.

$a_1T^2 + b_1T + c_1$ passes through $T = 22$ and $T = 42$.

$$a_1(22)^2 + b_1(22) + c_1 = 4181 \qquad (1)$$

$$a_1(42)^2 + b_1(42) + c_1 = 4179 \qquad (2)$$

$a_2T^2 + b_2T + c_2$ passes through $T = 42$ and $T = 52$.

$$a_2(42)^2 + b_2(42) + c_2 = 4179 \qquad (3)$$

$$a_2(52)^2 + b_2(52) + c_2 = 4186 \qquad (4)$$

$a_3T^2 + b_3T + c_3$ passes through $T = 52$ and $T = 82$.

$$a_3(52)^2 + b_3(52) + c_3 = 4186 \qquad (5)$$

$$a_3(82)^2 + b_3(82) + c_3 = 4199 \qquad (6)$$

$a_4T^2 + b_4T + c_4$ passes through $T = 82$ and $T = 100$.

$$a_4(82)^2 + b_4(82) + c_4 = 4199 \qquad (7)$$

$$a_4(100)^2 + b_4(100) + c_4 = 4217 \qquad (8)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $T = 42$

$$2a_1(42) + b_1 - 2a_2(42) - b_2 = 0 \qquad (9)$$

At $T = 52$

$$2a_2(52) + b_2 - 2a_3(52) - b_3 = 0 \qquad (10)$$

At $T = 82$

$$2a_3(82) + b_3 - 2a_4(82) - b_4 = 0 \qquad (11)$$

Assuming the first spline $a_1T^2 + b_1T + c_1$ is linear,

$$a_1 = 0 \qquad (12)$$

$$\begin{bmatrix} 484 & 22 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1764 & 42 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1764 & 42 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2704 & 52 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2704 & 52 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6724 & 82 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6724 & 82 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 100 & 1 \\ 84 & 1 & 0 & -84 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 104 & 1 & 0 & -104 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 164 & 1 & 0 & -164 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \end{bmatrix} = \begin{bmatrix} 4181 \\ 4179 \\ 4179 \\ 4186 \\ 4186 \\ 4199 \\ 4199 \\ 4217 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 12 equations gives the 12 unknowns as

i	a_i	b_i	c_i
1	0	–0.1	4183.2
2	0.08	–6.82	4324.3
3	–0.035556	5.1978	4011.9
4	0.090741	–15.515	4861.1

Therefore, the splines are given by

$$\begin{aligned} C_p(T) &= -0.1T + 4183.2, & 22 \le T \le 42 \\ &= 0.08T^2 - 6.82T + 4324.3, & 42 \le T \le 52 \\ &= -0.035556T^2 + 5.1978T + 4011.9, & 52 \le T \le 82 \\ &= 0.090741T^2 - 15.515T + 4861.1, & 82 \le T \le 100 \end{aligned}$$

At $T = 61$

$$\begin{aligned} C_p(61) &= -0.035556(61)^2 + 5.1978(61) + 4011.9 \\ &= 4196.6 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$\begin{aligned} |\in_a| &= \left| \frac{4196.6 - 4189.9}{4196.6} \right| \times 100 \\ &= 0.16013\% \end{aligned}$$

b) A more accurate calculation of the heat required to boil the water is given by

$$Q = m \int_{T_r}^{T_b} C_p dT$$

$T_r =$ room temperature $({}^\circ\text{C})$

$T_b =$ boiling temperature of water $(°\text{C})$

Given

$$T_r = 22\,°\text{C}$$
$$T_b = 100\,°\text{C}$$

To find $\int_{T_r}^{T_b} C_p dT$, we can integrate the quadratic splines with respect to temperature.

$$\int_{T_r}^{T_b} C_p dT = \int_{22}^{100} C_p(T)dT$$

$$= \int_{22}^{42} C_p(T)dT + \int_{42}^{52} C_p(T)dT + \int_{52}^{82} C_p(T)dT + \int_{82}^{100} C_p(T)dT$$

$$= \int_{22}^{42}(-0.1T + 4183.2)dT + \int_{42}^{52}(0.08T^2 - 6.82T + 4324.3)dT$$

$$+ \int_{52}^{82}(-0.035556T^2 + 5.1978T + 4011.9)dT$$

$$+ \int_{82}^{100}(0.090741T^2 - 15.515T + 4861.1)dT$$

$$= \left[-0.1\frac{T^2}{2} + 4183.2T\right]_{22}^{42} + \left[0.08\frac{T^3}{3} - 6.82\frac{T^2}{2} + 4324.3T\right]_{42}^{52}$$

$$+ \left[-0.035556\frac{T^3}{3} + 5.1978\frac{T^2}{2} + 4011.9T\right]_{52}^{82} + \left[0.090741\frac{T^3}{3} - 15.515\frac{T^2}{2} + 4861.1T\right]_{82}^{100}$$

$$= [83600] + [41812] + [125940] + [75656]$$

$$= 3.2700 \times 10^5 \ \frac{\text{J}}{\text{kg}}$$

To compare this result with our results from part (a), we take the average specific heat over this interval, given by

$$C_{p,avg} = \frac{\int_{T_r}^{T_b} C_p dT}{T_b - T_r}$$

$$= \frac{3.2700 \times 10^5}{100 - 22}$$

$$= 4192.3 \frac{\text{J}}{\text{kg} - °\text{C}}$$

Knowing that part (b) is the more accurate way of calculating the heat transfer, we define the absolute relative approximate error $|\epsilon_a|$ by

$$\begin{aligned}\left|\in_a\right| &= \left|\frac{4192.3-4196.6}{4192.3}\right|\times 100\\ &= 0.10211\%\end{aligned}$$