Chapter 05.04
Lagrangian Interpolation – More Examples
Chemical Engineering

Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at $61°\text{C}$. The specific heat of water is given as a function of time in Table 1.

Table 1 Specific heat of water as a function of temperature.

Temperature, T $(°\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - °\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

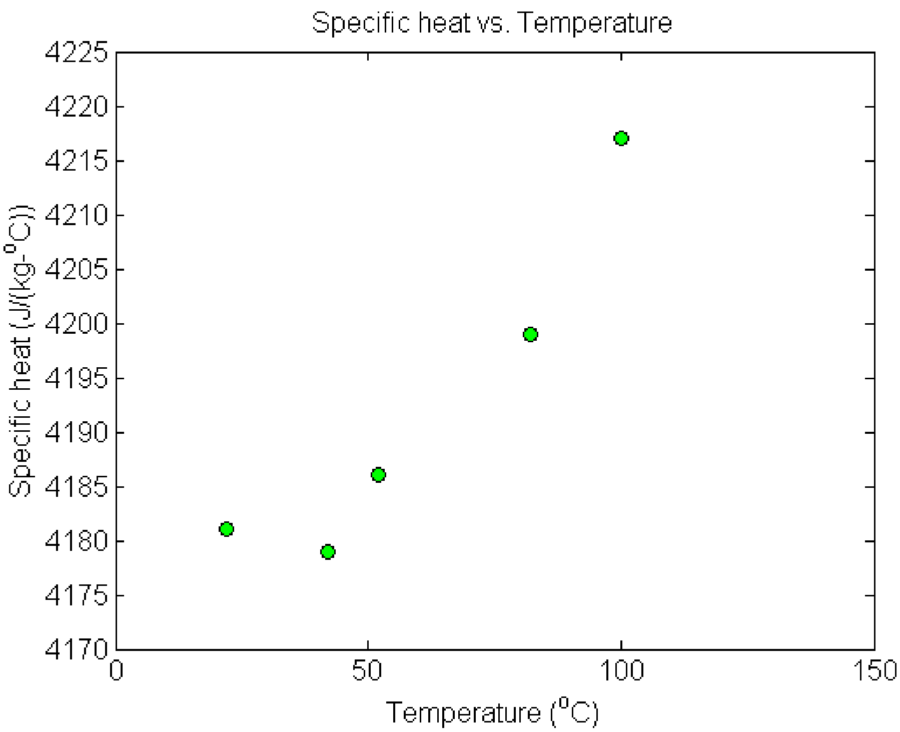

Figure 1 Specific heat of water vs. temperature.

Determine the value of the specific heat at $T = 61°\text{C}$ using a first order Lagrange polynomial.

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the specific heat is given by

$$\begin{aligned} C_p(T) &= \sum_{i=0}^{1} L_i(T) C_p(T_i) \\ &= L_0(T) C_p(T_0) + L_1(T) C_p(T_1) \end{aligned}$$

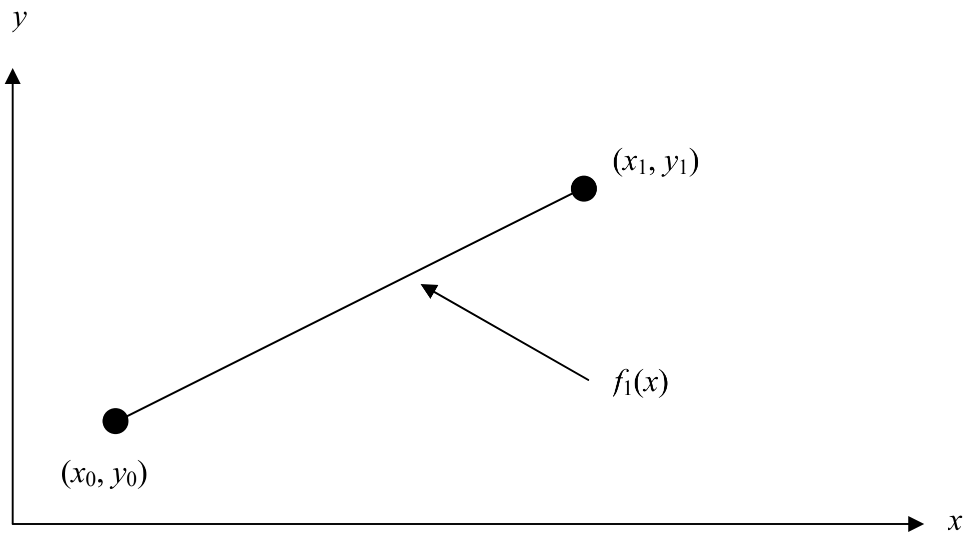

Figure 2 Linear interpolation.

Since we want the velocity at $T = 61°\text{C}$, we need to choose the two data points that are closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$ to evaluate it. The two points are $T_0 = 52$ and $T_1 = 82$.

Then

$$T_0 = 52,\ C_p(T_0) = 4186$$
$$T_1 = 82,\ C_p(T_1) = 4199$$

gives

$$L_0(T) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{T - T_j}{T_0 - T_j} = \frac{T - T_1}{T_0 - T_1}$$

$$L_1(T) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{T - T_j}{T_1 - T_j} = \frac{T - T_0}{T_1 - T_0}$$

Hence

$$C_p(T) = \frac{T - T_1}{T_0 - T_1} C_p(T_0) + \frac{T - T_0}{T_1 - T_0} C_p(T_1)$$
$$= \frac{T - 82}{52 - 82}(4186) + \frac{T - 52}{82 - 52}(4199),\ 52 \le T \le 82$$
$$C_p(61) = \frac{61 - 82}{52 - 82}(4186) + \frac{61 - 52}{82 - 52}(4199)$$

$$= 0.7(4186) + 0.3(4199)$$

$$= 4189.9 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}}$$

You can see that $L_0(T) = 0.7$ and $L_1(T) = 0.3$ are like weightages given to the specific heats at $T = 52$ and $T = 82$ to calculate the specific heat at $T = 61$.

Example 2

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C. The specific heat of water is given as a function of time in Table 2.

Table 2 Specific heat of water as a function of temperature.

Temperature, T $({}^\circ\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - {}^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61^\circ\text{C}$ using a second order Lagrange polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the specific heat given by

$$C_p(T) = \sum_{i=0}^{2} L_i(T) C_p(T_i)$$

$$= L_0(T) C_p(T_0) + L_1(T) C_p(T_1) + L_2(T) C_p(T_2)$$

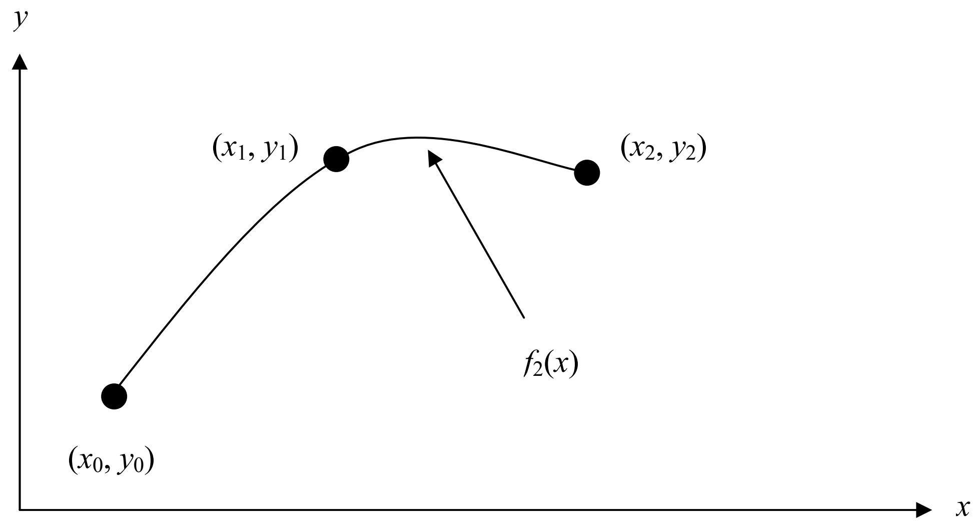

Figure 3 Quadratic interpolation.

Since we want to find the specific heat at $T = 61°\text{C}$, we need to choose the three data points that are closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$ to evaluate it. The three points are $T_0 = 42,\ T_1 = 52$ and $T_2 = 82$.

Then

$$T_0 = 42,\ C_p(T_0) = 4179$$
$$T_1 = 52,\ C_p(T_1) = 4186$$
$$T_2 = 82,\ C_p(T_2) = 4199$$

gives

$$L_0(T) = \prod_{\substack{j=0\\ j\neq 0}}^{2} \frac{T - T_j}{T_0 - T_j}$$
$$= \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)$$
$$L_1(T) = \prod_{\substack{j=0\\ j\neq 1}}^{2} \frac{T - T_j}{T_1 - T_j}$$
$$= \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)$$
$$L_2(T) = \prod_{\substack{j=0\\ j\neq 2}}^{2} \frac{T - T_j}{T_2 - T_j}$$
$$= \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)$$

Hence

$$C_p(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)C_p(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)C_p(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)C_p(T_2),$$

$$T_1 \le T \le T_2$$

$$C_p(61) = \frac{(61-52)(61-82)}{(42-52)(42-82)}(4179) + \frac{(61-42)(61-82)}{(52-42)(52-82)}(4186) + \frac{(61-42)(61-52)}{(82-42)(82-52)}(4199)$$

$$= (-0.4725)(4179) + (1.33)(4186) + (0.1425)(4199)$$

$$= 4191.2 \frac{\text{J}}{\text{kg} - °\text{C}}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{4191.2 - 4189.9}{4191.2}\right| \times 100$$

$$= 0.030063\%$$

Example 3

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at $61°\text{C}$. The specific heat of water is given as a function of time in Table 3.

Table 3 Specific heat of water as a function of temperature.

Temperature, T $(°\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - °\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61°\text{C}$ using a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

Solution

For third order Lagrange polynomial interpolation (also called cubic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^{3} L_i(T)C_p(T_i)$$

$$= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2) + L_3(T)C_p(T_3)$$

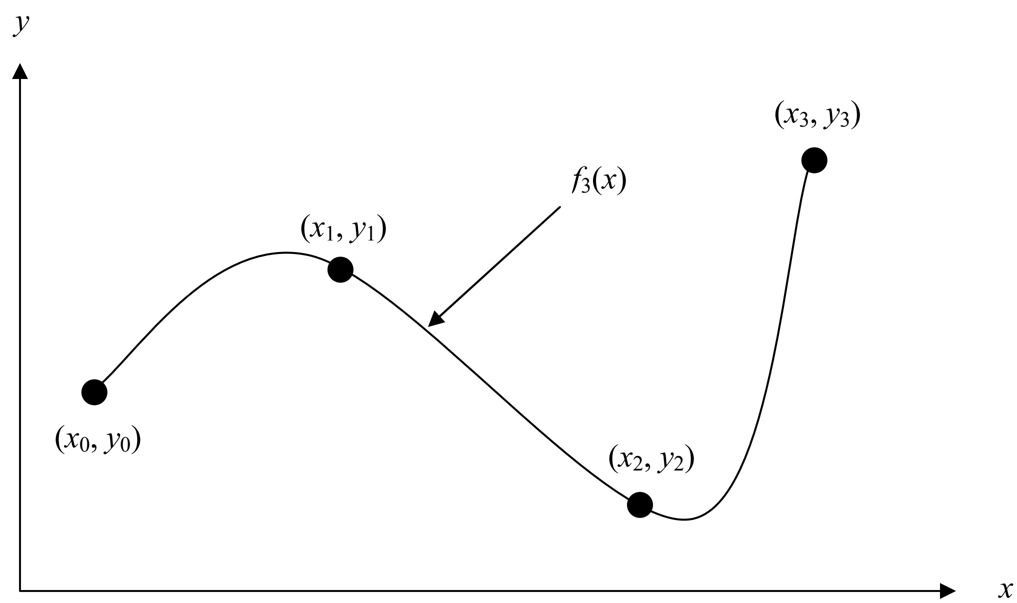

Figure 4 Cubic interpolation.

Since we wish to find the velocity at $T = 61°\text{C}$, we need to choose four data points that are closest to $T = 61°\text{C}$ and bracket $T = 61°\text{C}$ to evaluate it. The four data points are $T_0 = 42$, $T_1 = 52$, $T_2 = 82$ and $T_3 = 100$. (Choosing the four points as $T_0 = 22$, $T_1 = 42$, $T_2 = 52$ and $T_3 = 82$ is equally valid.)

Then

$$T_0 = 42,\ C_p(T_0) = 4179$$
$$T_1 = 52,\ C_p(T_1) = 4186$$
$$T_2 = 82,\ C_p(T_2) = 4199$$
$$T_3 = 100,\ C_p(T_3) = 4217$$

gives

$$L_0(T) = \prod_{\substack{j=0 \\ j\neq 0}}^{3} \frac{T - T_j}{T_0 - T_j}$$
$$= \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)$$
$$L_1(T) = \prod_{\substack{j=0 \\ j\neq 1}}^{3} \frac{T - T_j}{T_1 - T_j}$$
$$= \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j\neq 2}}^{3} \frac{T - T_j}{T_2 - T_j}$$

$$= \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)$$

$$L_3(T) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{T - T_j}{T_3 - T_j}$$

$$= \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)$$

Hence

$$C_p(T) = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)C_p(T_0) + \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)C_p(T_1)$$

$$+ \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)C_p(T_2) + \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)C_p(T_3)$$

$$T_0 \le T \le T_3$$

$$v(16) = \frac{(61-52)(61-82)(61-100)}{(42-52)(42-82)(42-100)}(4179) + \frac{(61-42)(61-82)(61-100)}{(52-42)(52-82)(52-100)}(4186)$$

$$+ \frac{(61-42)(61-52)(61-100)}{(82-42)(82-52)(82-100)}(4199) + \frac{(61-42)(61-52)(61-82)}{(100-42)(100-52)(100-82)}(4217)$$

$$= (-0.31772)(4179) + (1.0806)(4186) + (0.30875)(4199) + (-0.071659)(4217)$$

$$= 4190.0 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{4190.0 - 4191.2}{4190.0}\right| \times 100$$

$$= 0.027295\%$$

INTERPOLATION	
Topic	Lagrange Interpolation
Summary	Examples of the Lagrangian method of interpolation.
Major	Chemical Engineering
Authors	Autar Kaw
Date	November 23, 2009
Web Site	http://numericalmethods.eng.usf.edu