Chapter 02.03
Differentiation of Discrete Functions-More Examples
Chemical Engineering

Example 1

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 1 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,

(a) Compute the rate at which the radius of the drop was changing at $t = 2$ seconds.
(b) Estimate the rate at which the area of the contaminant was spreading across the pond at $t = 2$ seconds.

Table 1 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use the forward divided difference approximation of the first derivative to solve the above problem. Use a time step of 0.5 seconds.

Solution

(a) $$R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t}$$

$$t_i = 2$$

$$t_{i+1} = 2.5$$

$$\Delta t = t_{i+1} - t_i$$
$$= 2.5 - 2$$
$$= 0.5$$

$$R'(2) \approx \frac{R(2.5) - R(2)}{0.5}$$
$$= \frac{2.635 - 1.886}{0.5}$$
$$= 1.498 \text{ m/s}$$

(b) Area $= \pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A $\left(\text{m}^2\right)$	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

$$A'(t_i) \approx \frac{A(t_{i+1}) - A(t_i)}{\Delta t}$$

$$t_i = 2$$

$$t_{i+1} = 2.5$$

$$\begin{aligned}\Delta t &= t_{i+1} - t_i \\ &= 2.5 - 2 \\ &= 0.5\end{aligned}$$

$$\begin{aligned}A'(10) &\approx \frac{A(2.5) - A(2)}{0.5} \\ &= \frac{21.813 - 11.175}{0.5} \\ &= 21.276\ \text{m}^2/\text{s}\end{aligned}$$

Example 2

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 2 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,

(a) Compute the rate at which the radius of the drop was changing at $t = 2$ seconds.

(b) Estimate the rate at which the area of the contaminant was spreading across the pond at $t = 2$ seconds.

Table 2 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use a third order polynomial interpolant for the radius and area calculations.

Solution

(a) For third order polynomial interpolation (also called cubic interpolation), we choose the radius given by

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the radius at $t = 2$, and we are using a third order polynomial, we need to choose the four points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.

The four points are $t_0 = 1.0$, $t_1 = 1.5$, $t_2 = 2.0$, and $t_3 = 2.5$. (Note: Choosing $t_0 = 1.5$, $t_1 = 2.0$, $t_2 = 2.5$, and $t_3 = 3.0$ is equally valid.)

$$t_0 = 1.0, \quad R(t_0) = 0.667$$
$$t_1 = 1.5, \quad R(t_1) = 1.225$$
$$t_2 = 2.0, \quad R(t_2) = 1.886$$
$$t_3 = 2.5, \quad R(t_3) = 2.635$$

such that

$$R(1.0) = 0.667 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3$$
$$R(1.5) = 1.225 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3$$
$$R(2.0) = 1.886 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3$$
$$R(2.5) = 2.635 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 2 & 4 & 8 \\ 1 & 2.5 & 6.25 & 15.625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.667 \\ 1.225 \\ 1.886 \\ 2.635 \end{bmatrix}$$

Solving the above gives

$$a_0 = -0.08$$
$$a_1 = 0.471$$
$$a_2 = 0.296$$
$$a_3 = -0.02$$

Hence

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= -0.08 + 0.471t + 0.296t^2 - 0.02t^3, \quad 1 \le t \le 2.5$$

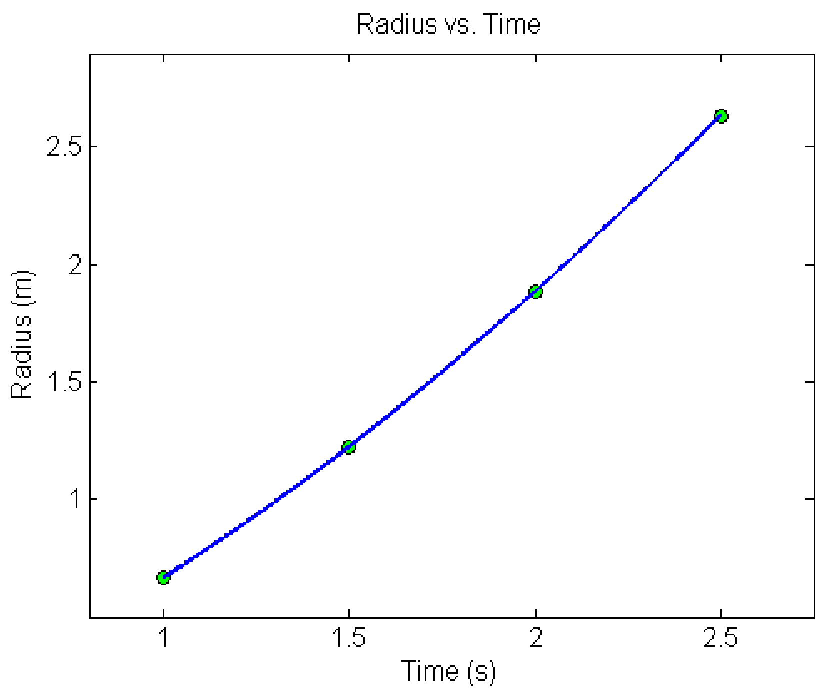

Figure 1 Graph of radius vs. time.

The derivative of the radius at $t = 2$ is given by

$$R'(2) = \frac{d}{dt}R(t)\Big|_{t=2}$$

Given that $R(t) = -0.08 + 0.471t + 0.296t^2 - 0.02t^3, \quad 1 \le t \le 2.5$,

$$\begin{aligned} R'(t) &= \frac{d}{dt}R(t) \\ &= \frac{d}{dt}\left(-0.08 + 0.471t + 0.296t^2 - 0.02t^3\right) \\ &= 0.471 + 0.592t - 0.06t^2, \quad 1 \le t \le 2.5 \end{aligned}$$

$$\begin{aligned} R'(2) &= 0.471 + 0.592(2) - 0.06(2)^2 \\ &= 1.415 \text{ m/s} \end{aligned}$$

(b) Area $= \pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A (m^2)	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

For third order polynomial interpolation (also called cubic interpolation), we choose the area given by

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the area at $t = 2$, and we are using a third order polynomial, we need to choose the four points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.

The four points are $t_0 = 1.0$, $t_1 = 1.5$, $t_2 = 2.0$ and $t_3 = 2.5$. (Note: Choosing $t_0 = 1.5$, $t_1 = 2.0$, $t_2 = 2.5$, and $t_3 = 3.0$ is equally valid.)

$$t_0 = 1.0, \quad A(t_0) = 1.3977$$
$$t_1 = 1.5, \quad A(t_1) = 4.7144$$
$$t_2 = 2.0, \quad A(t_2) = 11.175$$
$$t_3 = 2.5, \quad A(t_3) = 21.813$$

such that

$$A(1.0) = 1.3977 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3$$
$$A(1.5) = 4.7144 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3$$
$$A(2.0) = 11.175 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3$$
$$A(2.5) = 21.813 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 2 & 4 & 8 \\ 1 & 2.5 & 6.25 & 15.625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.3977 \\ 4.7144 \\ 11.175 \\ 21.813 \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.057900$$
$$a_1 = -0.12075$$
$$a_2 = 0.081468$$
$$a_3 = 1.3790$$

Hence

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= 0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3, \quad 1 \le t \le 2.5$$

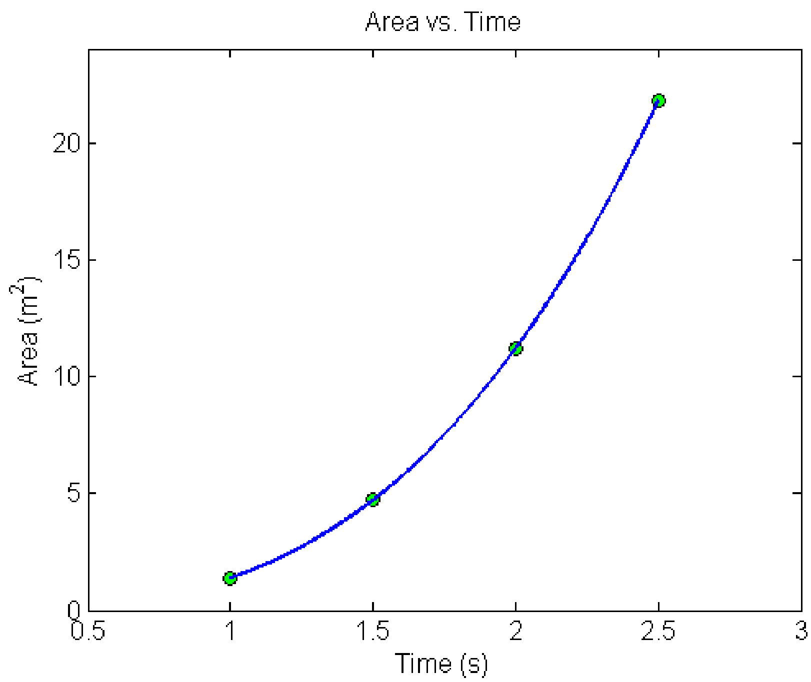

Figure 2 Graph of area vs. time.

The derivative of the area at $t=2$ is given by

$$A'(2)=\frac{d}{dt}A(t)\Big|_{t=2}$$

Given that $A(t)=0.057900-0.12075t+0.081468t^2+1.3790t^3,\quad 1\le t\le 2.5$,

$$\begin{aligned}A'(t)&=\frac{d}{dt}A(t)\\&=\frac{d}{dt}\left(0.057900-0.12075t+0.081468t^2+1.3790t^3\right)\\&=-0.12075+0.16294t+4.1371t^2,\quad 1\le t\le 2.5\end{aligned}$$

$$\begin{aligned}A'(2)&=-0.12075+0.16294(2)+4.1371(2)^2\\&=16.754\ \text{m}^2/\text{s}\end{aligned}$$

Example 3

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 3 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,

(c) Compute the rate at which the radius of the drop was changing at $t=2$ seconds.

(d) Estimate the rate at which the area of the contaminant was spreading across the pond at $t=2$ seconds.

Table 3 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use second order Lagrangian polynomial interpolation to solve the problem.

Solution

(a) For second order Lagrangian polynomial interpolation, we choose the radius given by

$$R(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)R(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)R(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)R(t_2)$$

Since we want to find the radius at $t=2$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t=2$ that also bracket $t=2$ to evaluate it.

The three points are $t_0=1.5$, $t_1=2.0$, and $t_2=2.5$.

Differentiating the above equation gives

$$R'(t)=\frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}R(t_0)+\frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}R(t_1)+\frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}R(t_2)$$

Hence

$$\begin{aligned}R'(2)&=\frac{2(2)-(2.0+2.5)}{(1.5-2.0)(1.5-2.5)}(1.225)+\frac{2(2)-(1.5+2.5)}{(2.0-1.5)(2.0-2.5)}(1.886)\\&\quad+\frac{2(2)-(1.5+2.0)}{(2.5-1.5)(2.5-2.0)}(2.635)\\&=1.41\ \text{m/s}\end{aligned}$$

(b) Area $=\pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A (m^2)	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

For second order Lagrangian polynomial interpolation, we choose the area given by

$$A(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)A(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)A(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)A(t_2)$$

Since we want to find the area at $t=2$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t=2$ that also bracket $t=2$ to evaluate it. The three points are $t_0=1.5$, $t_1=2.0$, and $t_2=2.5$.

Differentiating the above equation gives

$$A'(t)=\frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}A(t_0)+\frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}A(t_1)+\frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}A(t_2)$$

Hence

$$A'(2)=\frac{2(2)-(2.0+2.5)}{(1.5-2.0)(1.5-2.5)}(4.7144)+\frac{2(2)-(1.5+2.5)}{(2.0-1.5)(2.0-2.5)}(11.175)$$

$$+\frac{2(2)-(1.5+2.0)}{(2.5-1.5)(2.5-2.0)}(21.813)$$
$$=17.099\ \text{m}^2/\text{s}$$

DIFFERENTIATION	
Topic	Discrete Functions-More Examples
Summary	Examples of Discrete Functions
Major	Chemical Engineering
Authors	Autar Kaw
Date	August 7, 2009
Web Site	http://numericalmethods.eng.usf.edu