Integration Using the Gauss Quadrature Rule - Convergence

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NOTE: This worksheet demonstrates the use of Maple to illustrate the Gauss Quadrature rule of integration.

Introduction

Gauss Quadrature Rule is another method of estimating an integral. The theory behind the two point Gauss Quadrature Rule is to approximate the integral by taking the area under a straight line connecting any two points on the curve that are not predetermined as $a$ and $b$, but as unknowns $x_1$ and $x_2$. For $n$-points rules, the general form to approximate the integral is

$$
\int_{a}^{b} f(x) \, dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots + c_n f(x_n)
$$

where $c_i$ and $x_i$ are the weighting factors and function arguments used in Gauss Quadrature formulas, respectively. However, these factors and arguments are already defined to approximate any integral from $-1$ to $1$. To be able to use them, the limits of the integral of the function $f(x)$ need to be changed to $[-1,1]$.

$$
\int_{-1}^{1} f(x) \, dx = \int_{\frac{b-a}{2}}^{\frac{b-a}{2}} f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right) \, dx
$$

NOTE: Weighting factors $c$ and function arguments $x$ used in Gauss Quadrature Rule have already been defined in the textbook for up to six points.

The following procedure will illustrate the Gauss Quadrature Rule of integration. The user may enter any function $f(x)$, the lower and upper limit for the function, and the number of points $n$ in the data section (up to six points). By entering this data, the program will calculate the exact value of the integral, followed by the results using the Gauss Quadrature Rule with $n$ points. The program will also display the true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error, and the number of significant digits that are at least correct.

```
> restart;
```

Section I: Input Data
The following is the data that is used to solve the integral using the Gauss Quadrature rule with \( n \) points.

The integrand:

\[
f := x \rightarrow \frac{300 \times x}{1 + e^x}
\]

The lower limit of the integral:

\[
a := 0.0;
\]

The upper limit of the integral:

\[
b := 10.0;
\]

The number of points \( n \):

\[
n := 6
\]

This is the end of the user's section. All information must be entered before proceeding to the next section.

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**Section II: Procedure**

The following procedure determines the approximate value of the integral with \( n \) points.

```maple
Gauss := proc(n, a, b, f)
local AV, C, X, f_new, sum, i;

The weighting factors for Gauss Quadrature rule for \( n \) points (up to six points)
C := array(1 .. 6, 1 .. 6):
C[1, 1] := 2:
C[1, 2] := 1; C[2, 2] := 1:
C[1, 3] := 0.555555556: C[2, 3] := 0.888888889: C[3, 3] := 0.555555556:

The function arguments for Gauss quadrature rule for \( n \) points (up to six points)
X := array(1 .. 6, 1 .. 6):
X[1, 1] := 0:
X[1, 2] := -0.577350269: X[2, 2] := 0.577350269:
```

X[1, 3] := -0.774596669: X[2, 3] := 0: X[3, 3] := 0.774596669:

f_new := x -> f((b-a)/2*x+(b+a)/2)*(b-a)/2:

sum := 0:
for i from 1 by 1 to n do
    sum := sum + C[i, n]*f_new(X[i, n]):
end do:

AV := sum:
return (AV):
end proc:

Section III: Calculation

> plot(f(x), x=a..b, title="f(x) vs x", thickness=3, color=black);
The exact value of the integral (EV):

```
> EV:=evalf(int(f(x),x=a..b)):
> for i from 1 by 1 to n do
    AV is the average value of the integral using n points
    AV[i]:=Gauss(i,a,b,f):
    Et is the true error
    Et[i]:=EV-AV[i]:
    abs_et is the absolute relative true percentage error
    abs_et[i]:=abs(Et[i]/EV)*100.0:
    if (i>1) then
        Ea is the approximate error
        Ea[i]:=AV[i]-AV[i-1]:
        ea is the absolute approximate relative percentage error
        ea[i]:=abs(Ea[i]/AV[i])*100.0:
        sig is the least correct significant digits
        sig[i]:=floor((2-log10(ea[i]/0.5))):
```
if sig[i]<0 then
    sig[i]:=0:
end if:
end if:
end do:

Section IV: Spreadsheet

```maple
> with( Spread ):
> EvaluateSpreadsheet(Gauss):

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The number of points</td>
<td>Exact Value</td>
<td>Approximate Value</td>
<td>True Er.</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>246.5903</td>
<td>100.3928</td>
<td>146.19</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>246.5903</td>
<td>346.2051</td>
<td>-99.61</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>246.5903</td>
<td>275.4838</td>
<td>-28.89</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>246.5903</td>
<td>243.9871</td>
<td>2.603</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>246.5903</td>
<td>245.3995</td>
<td>1.190</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>246.5903</td>
<td>246.6540</td>
<td>-0.063</td>
</tr>
</tbody>
</table>
```

Section V: Graphs

```maple
> with(plots):
Warning, the name changecoords has been redefined

> data:=[seq([i,AV[i]],i=1..n)]:
> pointplot(data,connect=true,color=red,axes=boxed,title="Approximate value of the integral as a function of the number of points",axes=BOXED,labels=['number of points','AV'],thickness=3);

> data:=[seq([i,Et[i]],i=1..n)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="True error as a function of the number of points",axes=BOXED,labels=['number of points','Et'],thickness=3);
```
> data:=[seq([i,abs_et[i]],i=1..n)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="Absolute relative true percentage error as a function of the number of points",axes=BOXED,labels=["number of points","abs_et"],thickness=3);

> data:=[seq([i,Ea[i]],i=2..n)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Approximate error as a function of the number of points",axes=BOXED,labels=["number of points","Ea"],thickness=3);

> data:=[seq([i,ea[i]],i=2..n)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Absolute approximate relative percentage error as a function of the number of points",axes=BOXED,labels=["number of points","ea"],thickness=3);

> data:=[seq([i,sig[i]],i=2..n)]:
> pointplot(data,connect=true,color=brown,axes=boxed,title="The least correct significant digits as a function of the number of points",axes=BOXED,labels=["number of points","sig"],thickness=3);

Approximate value of the integral as a function of the number of points
Absolute approximate relative percentage error as a function of the number of points

The least correct significant digits as a function of the number of points