Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to
1. develop Runge-Kutta 4th order method for solving ordinary differential equations,
2. find the effect size of step size has on the solution,
3. know the formulas for other versions of the Runge-Kutta 4th order method

What is the Runge-Kutta 4th order method?
Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form
\[
\frac{dy}{dx} = f(x, y), y(0) = y_0
\]
So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1
Rewrite
\[
\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5
\]
in
\[
\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}
\]
Solution

\[
\frac{dy}{dx} + 2y = 1.3e^{-x}, \quad y(0) = 5
\]
\[
\frac{dy}{dx} = 1.3e^{-x} - 2y, \quad y(0) = 5
\]

In this case

\[ f(x, y) = 1.3e^{-x} - 2y \]

Example 2

Rewrite

\[ e^x \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), \quad y(0) = 5 \]

in

\[ \frac{dy}{dx} = f(x, y), \quad y(0) = y_0 \text{ form.} \]

Solution

\[ e^x \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), \quad y(0) = 5 \]
\[ \frac{dy}{dx} = \frac{2 \sin(3x) - x^2 y^2}{e^x}, \quad y(0) = 5 \]

In this case

\[ f(x, y) = \frac{2 \sin(3x) - x^2 y^2}{e^x} \]

The Runge-Kutta 4th order method is based on the following

\[ y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \] \hspace{1cm} (1)

where knowing the value of \( y = y_i \) at \( x_i \), we can find the value of \( y = y_{i+1} \) at \( x_{i+1} \), and

\[ h = x_{i+1} - x_i \]

Equation (1) is equated to the first five terms of Taylor series

\[ y_{i+1} = y_i + \frac{dy}{dx}\bigg|_{x_i,y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2y}{dx^2}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3y}{dx^3}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^3 \]
\[ + \frac{1}{4!} \frac{d^4y}{dx^4}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^4 \] \hspace{1cm} (2)

Knowing that \( \frac{dy}{dx} = f(x, y) \) and \( x_{i+1} - x_i = h \)

\[ y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 + \frac{1}{3!} f''(x_i, y_i)h^3 + \frac{1}{4!} f'''(x_i, y_i)h^4 \] \hspace{1cm} (3)

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

\[ y_{i+1} = y_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h \] \hspace{1cm} (4)
Runge-Kutta 4th Order Method

\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h) \]
\[ k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h) \]
\[ k_4 = f(x_i + h, y_i + k_3h) \]

**Example 3**

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor, a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 \( \mu \)F, the ordinary differential equation to be solved is

\[
\frac{dv(t)}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left( \left| \frac{18 \cos(120\pi(t))}{0.04} \right| - 2 - v(t) \right) \right\}
\]

\[ v(0) = 0 \]

Using the Runge-Kutta 4th order method, find voltage across the capacitor at \( t = 0.00004 \) s. Use step size \( h = 0.00002 \) s.

**Solution**

\[
\frac{dv}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left( \left| \frac{18 \cos(120\pi(t))}{0.04} \right| - 2 - v \right) \right\}
\]

\[ f(t, v) = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left( \left| \frac{18 \cos(120\pi(t))}{0.04} \right| - 2 - v \right) \right\} \]

\[ v_{i+1} = v_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h \]

For \( i = 0, t_0 = 0, v_0 = 0 \)

\[ k_1 = f(t_0, v_0) \]
\[ = f(0,0) \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left( \left| \frac{18 \cos(120\pi(0))}{0.04} \right| - 2 - 0 \right) \right\} \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max(400,0) \right\} \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + 400 \right\} \]
\[ = 2.6660 \times 10^6 \]
\[k_2 = f\left(t_0 + \frac{1}{2} h, v_0 + \frac{1}{2} k_1 h\right)\]
\[= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(2.6660 \times 10^6)0.00002\right)\]
\[= f(0.00001, 26.660)\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max\left(\frac{18\cos(120\pi(0.00001)) - 2 - (26.660)}{0.04}, 0\right)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max(-266.50, 0)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 - 266.50\right\}\]
\[= -666.67\]
\[k_3 = f\left(t_0 + \frac{1}{2} h, v_0 + \frac{1}{2} k_2 h\right)\]
\[= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(-666.67)0.00002\right)\]
\[= f(0.00001, -0.006667)\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max\left(\frac{18\cos(120\pi(0.00001)) - 2 - (-0.006667)}{0.04}, 0\right)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max(400.16, 0)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + 400.16\right\}\]
\[= 2.6671 \times 10^6\]
\[k_4 = f\left(t_0 + h, v_0 + k_2 h\right)\]
\[= f\left(0 + 0.00002, 0 + (2.6671 \times 10^6)0.00002\right)\]
\[= f(0.00002, 53.342)\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max\left(\frac{18\cos(120\pi(0.00002)) - 2 - (53.342)}{0.04}, 0\right)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + \max(-933.56, 0)\right\}\]
\[= \frac{1}{150 \times 10^{-6}} \left\{-0.1 + 0\right\}\]
\[= -666.67\]
\[v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h\]
\[= 0 + \frac{1}{6}(2.6660 \times 10^6 + 2(-666.67) + 2(2.6671 \times 10^6) + (-666.67))0.00002\]
\[
\begin{align*}
\text{Runge-Kutta 4th Order Method} & \\
& = 0 + \frac{1}{6}(7.9982 \times 10^6)0.00002 \\
& = 26.661 \text{ V}
\end{align*}
\]

\(v_i\) is the approximate voltage at 
\(t = t_i = t_0 + h = 0 + 0.00002 = 0.00002\) 
\(v(0.00002) = v_i = 26.661 \text{ V}\)

For \(i = 1, t_i = 0.00002, v_i = 26.661\)

\(k_1 = f(t_1, v_1) = f(0.00002, 26.661)\)

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left[18\cos(120\pi(0.00002)) - 2 - (26.661), 0\right]\right\}
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.51, 0)\right\}
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + 0\right\}
\]

\[= -666.67\]

\(k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1h\right)\)

\[
= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)
\]

\[
= f(0.00003, 26.654)
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left[18\cos(120\pi(0.00003)) - 2 - (26.654), 0\right]\right\}
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.35, 0)\right\}
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + 0\right\}
\]

\[= -666.67\]

\(k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2h\right)\)

\[
= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)
\]

\[
= f(0.00003, 26.654)
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left[18\cos(120\pi(0.00003)) - 2 - (26.654), 0\right]\right\}
\]

\[
= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.35, 0)\right\}
\]
\[ k_4 = f(t_i + h, v_i + k_3 h) \]
\[ = f(0.00002 + (0.00002), 26.661 + (-666.67)0.00002) \]
\[ = f(0.00003, 26.647) \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left( \frac{18 \cos(120\pi(0.00003))}{0.04} \right) \right\} \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max(-265.87, 0) \right\} \]
\[ = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + 0 \right\} \]
\[ = -666.67 \]

\[ v_2 = v_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) h \]
\[ = 26.661 + \frac{1}{6} (-666.67 + 2(-666.67) + 2(-666.67) + (-666.67))0.00002 \]
\[ = 26.661 + \frac{1}{6} (-4000.0)0.00002 \]
\[ = 26.647 \text{ V} \]

\( v_2 \) is the approximate voltage at \( t = t_2 \)
\[ t_2 = t_i + h = 0.00002 + 0.00002 = 0.00004 \text{ s} \]
\[ v(.00004) \approx v_2 = 26.647 \text{ V} \]

Figure 1 compares the exact solution of \( v(0.00004) = 15.974 \text{ V} \) with the numerical solution using Runge-Kutta 4th order method step size of \( h = 0.00002 \text{ s} \).
Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at \( t = 0.00004 \) s.

**Table 1** Value of voltage at time, \( t = 0.00004 \) s for different step sizes.

| Step size, \( h \) | \( v(0.00004) \) | \( E_t \) | \( |\epsilon| \) % |
|-------------------|-----------------|---------|----------|
| 0.00004           | 53.335          | -37.361 | 233.89   |
| 0.00002           | 26.647          | -10.673 | 66.817   |
| 0.00001           | 15.986          | -0.012299 | 0.076996 |
| 0.000005          | 15.975          | -0.00050402 | 0.0031552 |
| 0.0000025         | 15.976          | -0.0015916 | 0.0099639 |
Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler’s method (Runge-Kutta 1st order method), Heun’s method (Runge-Kutta 2nd order method), and Runge-Kutta 4th order method.

The formula described in this chapter was developed by Runge. This formula is same as Simpson’s 1/3 rule, if \( f(x, y) \) were only a function of \( x \). There are other versions of the 4th order method just like there are several versions of the second order methods. The formula developed by Kutta is

\[
y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h
\]

where

\[
k_1 = f(x_i, y_i)
\]

\[
k_2 = f(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1)
\]

\[
k_3 = f(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2)
\]

\[
k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3)
\]

This formula is the same as the Simpson’s 3/8 rule, if \( f(x, y) \) is only a function of \( x \).
Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.