Fast Fourier Transform

Part: Informal Development of Fast Fourier Transform

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Lecture # 11

Chapter 11.05: Informal Development of Fast Fourier Transform

Major: All Engineering Majors

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Numerical Methods for STEM undergraduates

Informal Development of Fast Fourier Transform

Recall the DFT pairs of Equations (20) and (21) of Chapter 11.04 and swapping the indexes n, k one obtains

$$\tilde{C}_n = \sum_{k=0}^{N-1} f(k) e^{-in\left(w_0 = \frac{2\pi}{N}\right)k} \tag{1}$$

$$f(k) = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} \tilde{C}_n e^{in\left(w_0 = \frac{2\pi}{N}\right)k} \tag{2}$$

where $n, k = 0, 1, 2, 3, \ldots, N-1$ (3)

Let $E = e^{-i\frac{2\pi}{N}} \quad \left(hence \;\; E^N = e^{-i2\pi} = 1 \right)$ (4)

Informal Development cont.

Then Eq. (1) and Eq. (2) become

$$\tilde{C}_n = \tilde{C}(n) = \sum_{k=0}^{N-1} f(k) E^{nk} \tag{5}$$

$$f(k) = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} \tilde{C}_n E^{-nk}$$

Assuming $N = 4 = 2^{(r=2)}$, then

$$\left(\frac{1}{N}\right)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & E^{-1} & E^{-2} & E^{-3} \\ 1 & E^{-2} & E^{-4} & E^{-6} \\ 1 & E^{-3} & E^{-6} & E^{-9} \end{bmatrix} \begin{Bmatrix} \tilde{C}(0) \\ \tilde{C}(1) \\ \tilde{C}(2) \\ \tilde{C}(3) \end{Bmatrix} = \begin{Bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{Bmatrix} \tag{5A}$$

Informal Development cont.

To obtain the above unknown vector $\left\{\tilde{C}\right\}$ for a given vector $\left\{f\right\}$, the coefficient matrix can be easily converted as

$$\left[\left(\frac{1}{N}\right)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & E^{-1} & E^{-2} & E^{-3} \\ 1 & E^{-2} & E^{-4} & E^{-6} \\ 1 & E^{-3} & E^{-6} & E^{-9} \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & E^{1} & E^{2} & E^{3} \\ 1 & E^{2} & E^{4} & E^{6} \\ 1 & E^{3} & E^{6} & E^{9} \end{bmatrix}$$

Informal Development cont.

Hence, the unknown vector $\{\tilde{C}\}$ can be computed as with matrix vector operations, as following

$$\begin{Bmatrix} \tilde{C}(0) \\ \tilde{C}(1) \\ \tilde{C}(2) \\ \tilde{C}(3) \end{Bmatrix} = \begin{bmatrix} E^{(0)(0)} & E^{(0)(1)} & E^{(0)(2)} & E^{(0)(3)} \\ E^{(1)(0)} & E^{(1)(1)} & E^{(1)(2)} & E^{(1)(3)} \\ E^{(2)(0)} & E^{(2)(1)} & E^{(2)(2)} & E^{(2)(3)} \\ E^{(3)(0)} & E^{(3)(1)} & E^{(3)(2)} & E^{(3)(3)} \end{bmatrix} \begin{Bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{Bmatrix} \qquad (6)$$

Informal Development cont.

$$\begin{Bmatrix} \tilde{C}(0) \\ \tilde{C}(1) \\ \tilde{C}(2) \\ \tilde{C}(3) \end{Bmatrix} = \begin{bmatrix} E^0 & E^0 & E^0 & E^0 \\ E^0 & E^1 & E^2 & E^3 \\ E^0 & E^2 & E^4 & E^6 \\ E^0 & E^3 & E^6 & E^9 \end{bmatrix} \begin{Bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{Bmatrix} \qquad (7)$$

For $N = 4, k = 3$ and $n = 2$, then:

$$E^{nk} = E^6 = \left[E^{(N=4)}\right]E^2 = \left(e^{\frac{-i2\pi}{N}}\right)^N E^2 = \left[e^{-i2\pi}\right]E^2 = E^2$$

Informal Development cont.

Thus, in general (for $nk \geq N$)

$$E^{nk} = E^{U} \text{ where } U = \text{mod}(nk, N) \tag{8}$$

$$U = remainder\left(\frac{nk}{N}\right)$$

Informal Development cont.

Remarks:

a) Matrix times vector, shown in Eq. (7), will require 16 (or N^2) complex multiplications and 12 (or $N(N-1)$) complex additions.

b) Use of Eq. (8) will help to reduce the number of operation counts, as explained in the next section.

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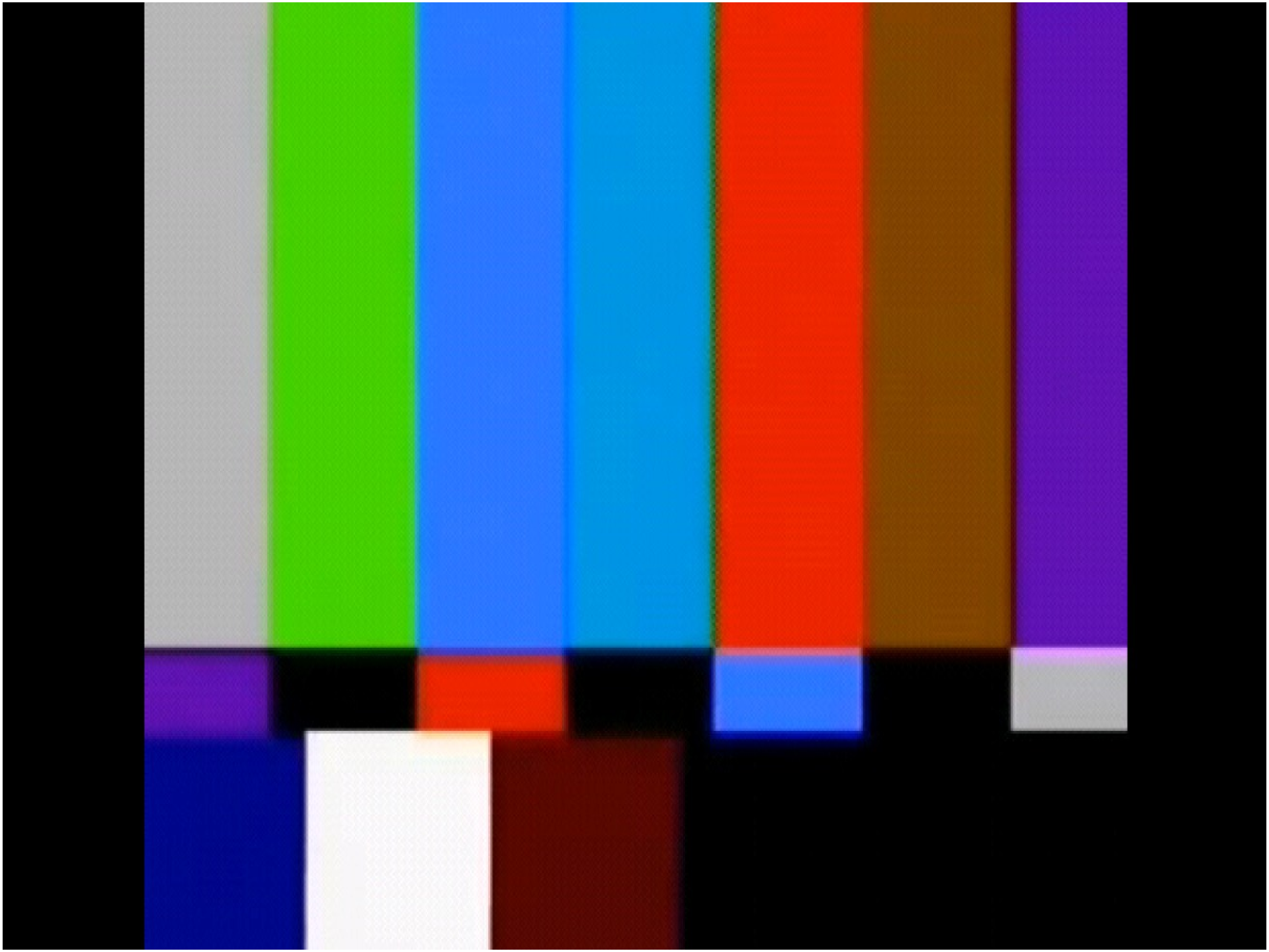

Fast Fourier Transform

Part: Factorized Matrix and Further Operation Count

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Lecture # 12

Chapter 11.05: Factorized Matrix and Further Operation Count (Contd.)

Equation (7) can be factorized as

$$\begin{Bmatrix} \tilde{C}(0) \\ \tilde{C}(2) \\ \tilde{C}(1) \\ \tilde{C}(3) \end{Bmatrix} = \begin{bmatrix} 1 & E^0 & 0 & 0 \\ 1 & E^2 & 0 & 0 \\ 0 & 0 & 1 & E^1 \\ 0 & 0 & 1 & E^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & E^0 & 0 \\ 0 & 1 & 0 & E^0 \\ 1 & 0 & E^2 & 0 \\ 0 & 1 & 0 & E^2 \end{bmatrix} \begin{Bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{Bmatrix} \qquad (9)$$

Let's define the following "inner – product"

$$\begin{Bmatrix} f_1(0) \\ f_1(1) \\ f_1(2) \\ f_1(3) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & E^0 & 0 \\ 0 & 1 & 0 & E^0 \\ 1 & 0 & E^2 & 0 \\ 0 & 1 & 0 & E^2 \end{bmatrix} \begin{Bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{Bmatrix} \qquad (10)$$

Factorized Matrix cont.

From Eq. (9) and (10) we obtain

$$f_1(0) = f(0) + E^0 f(2) \qquad (11A)$$

$$f_1(1) = f(1) + E^0 f(3) \qquad (11B)$$

$$f_1(2) = f(0) + E^2 f(2)$$

$$= f(0) - E^0 f(2) \qquad (11C)$$

with $$E^2 = e^{-i\frac{2\pi}{4}*2} = e^{-i\pi} = -1 = -E^0$$

$$f_1(3) = f(1) + E^2 f(3)$$

$$= f(1) - E^0 f(3) \qquad (11D)$$

Factorized Matrix cont.

Equations(11A through 11D) for the “inner” matrix times vector requires 2 complex multiplications and 4 complex additions.

Factorized Matrix cont.

Finally, performing the "outer" product (matrix times vector) on the RHS of Equation(9), one obtains

$$\begin{Bmatrix} \tilde{C}(0) \\ \tilde{C}(2) \\ \tilde{C}(1) \\ \tilde{C}(3) \end{Bmatrix} = \begin{Bmatrix} f_2(0) \\ f_2(1) \\ f_2(2) \\ f_2(3) \end{Bmatrix} = \begin{bmatrix} 1 & E^0 & 0 & 0 \\ 1 & E^2 & 0 & 0 \\ 0 & 0 & 1 & E^1 \\ 0 & 0 & 1 & E^3 \end{bmatrix} \begin{Bmatrix} f_1(0) \\ f_1(1) \\ f_1(2) \\ f_1(3) \end{Bmatrix} \quad (12)$$

Factorized Matrix cont.

$$f_2(0) = f_1(0) + E^0 f_1(1) \qquad (13A)$$

$$f_2(1) = f_1(0) + E^2 f_1(1) = f_1(0) - E^0 f_1(1) \qquad (13B)$$

$$f_2(2) = f_1(2) + E^1 f_1(3) \qquad (13C)$$

$$f_2(3) = f_1(2) + E^3 f_1(3) = f_1(2) + E^2 E^1 f_1(3)$$

$$= f_1(2) - E^1 f_1(3) \qquad (13D)$$

Factorized Matrix cont.

Again, Eqs (13A-13D) requires 2 complex multiplications And 4 complex additions. Thus, the complete RHS of Eq. (9) Can be computed by only 4 complex multiplications (or $N\frac{r}{2} = 4\frac{2}{2}$) and 8 complex additions (or $Nr = 4 * 2$),

where $N = 2^r$.

Since computational time is mainly controlled by the number of multiplications, implementing Eq. (9) will significantly reduce the number of multiplication operations, as compared to a direct matrix times vector operations. (as shown in Eq. (7)).

Factorized Matrix cont.

For a large number of data points,

$$Ratio = \frac{N^2}{\left(\frac{Nr}{2}\right)} = \left(\frac{2N}{r}\right) \qquad (14)$$

For $N = 2048 = 2^{(r=11)}$,Equation (14) gives:

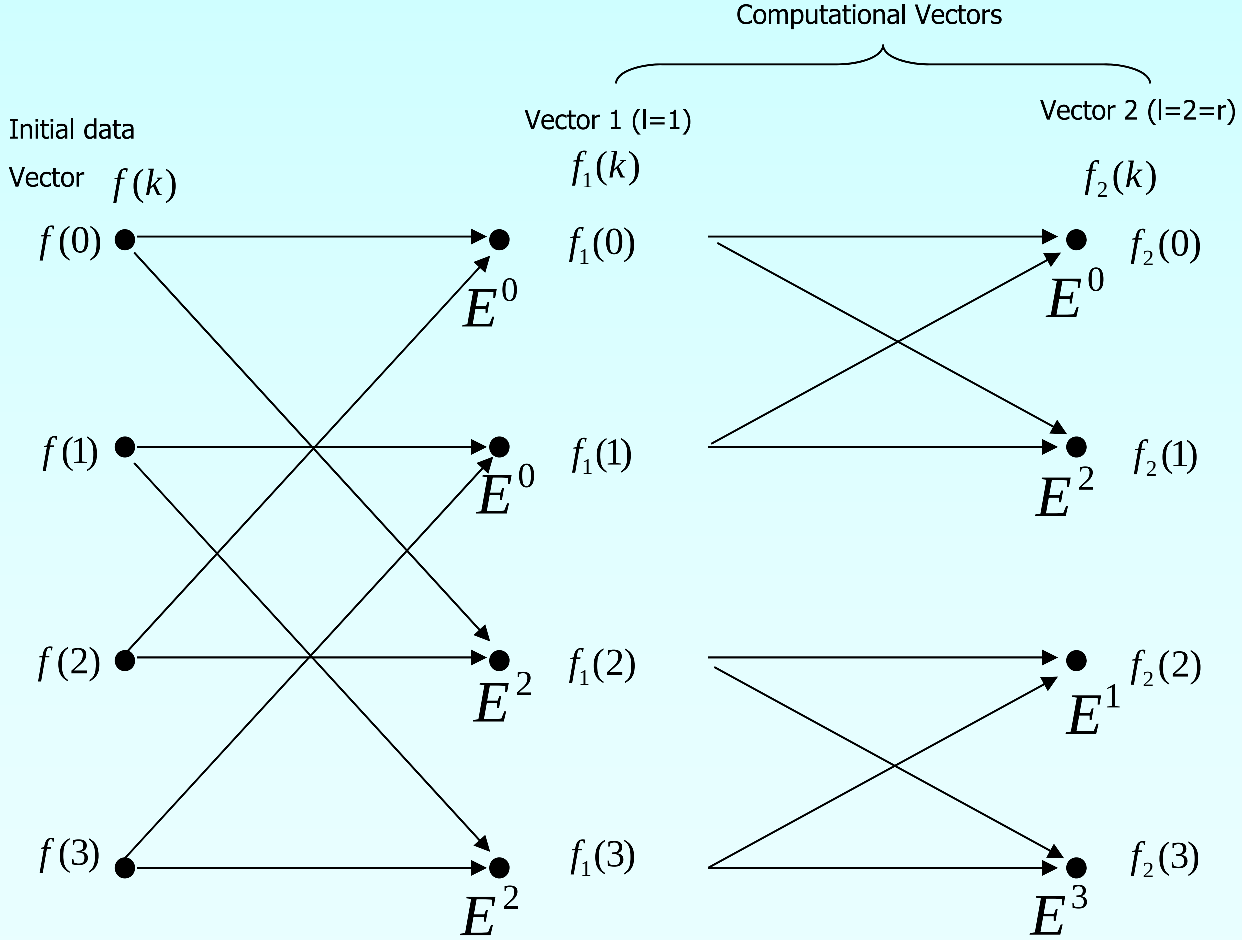
$$Ratio = \frac{2(2048)}{11} = 372.36$$

This implies that the number of complex multiplications involved in Eq. (9) is about 372 times less than the one involved in Eq. (7).

Graphical Flow of Eq. 9

- Consider the case $N = 2^r = 2^2 = 4$

Figure 1. Graphical form of FFT (Eq. 9) for the case $N = 2^r = 2^2 = 4$



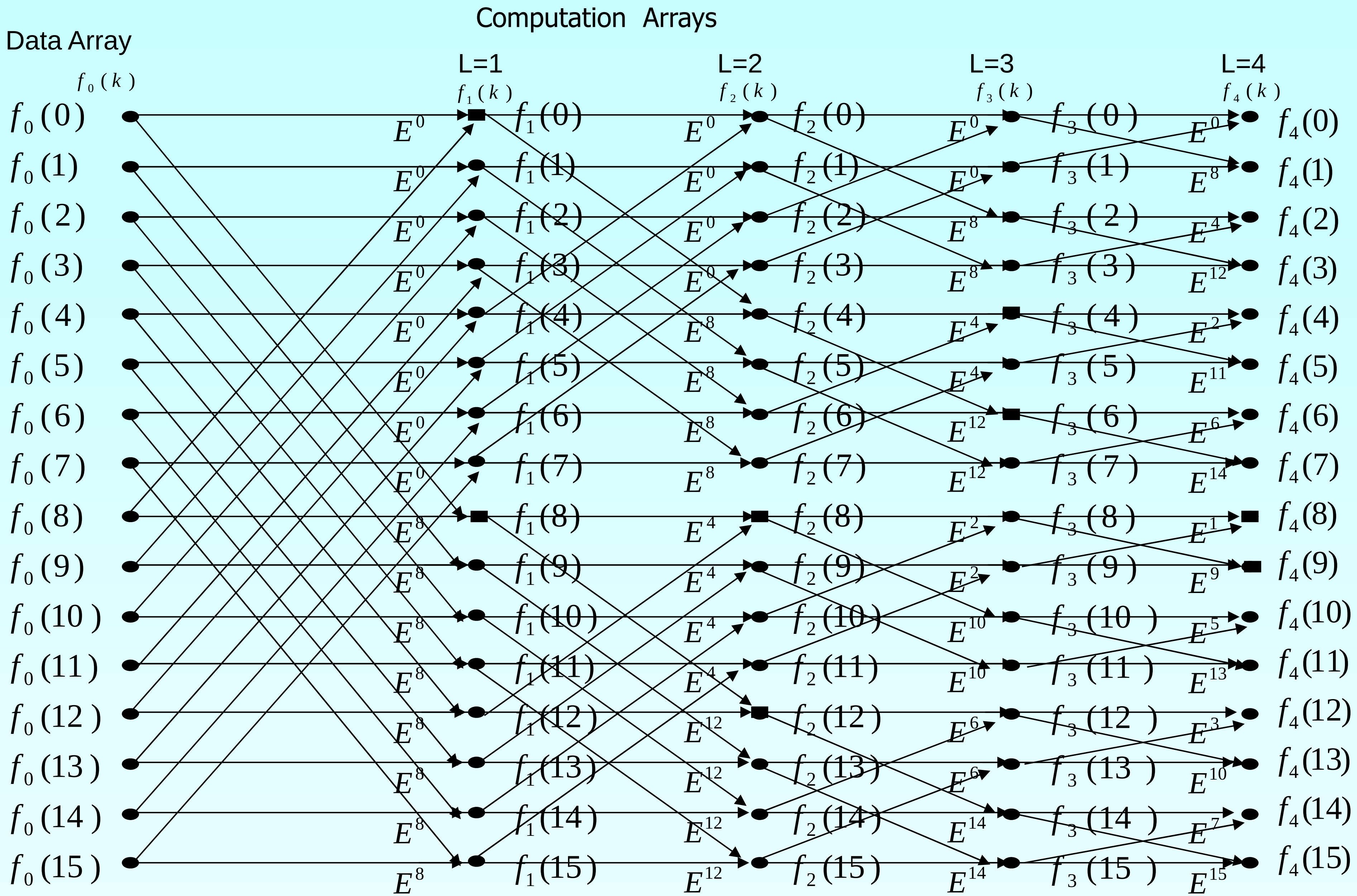


Figure 2. Graphical Form of FFT (Eq. 9) for the case

$$N = 2^r = 2^4 = 16$$

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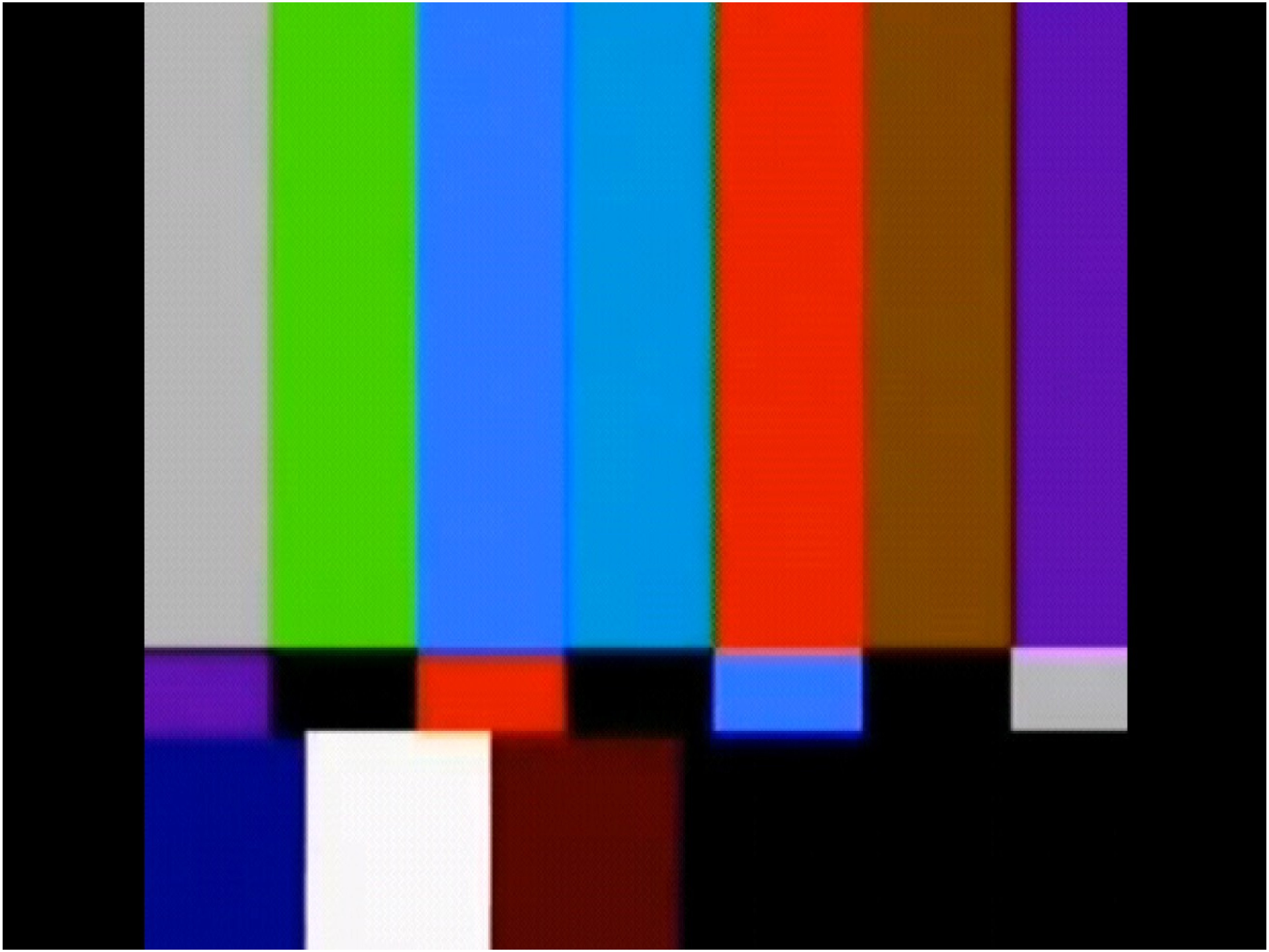

Fast Fourier Transform

Part: Companion Node Observation

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Lecture # 13

Chapter 11.05 : Companion Node Observation (Contd.)

Careful observation of Figure 2 has revealed that each computed l^{th} vector (where $l = 1,2,\ldots, r$ and $N = 2^r = 2^4 = 16$) we can always find two (companion) nodes which came from the same pair of nodes in the previous vector.

For example, $f_1(0)$ and $f_1(8)$ are computed in terms of $f(0)$ and $f(8)$.

Similarly, the companion nodes $f_2(8)$ and $f_2(12)$ are computed from the same pair of nodes as $f_1(8)$ and $f_1(12)$.

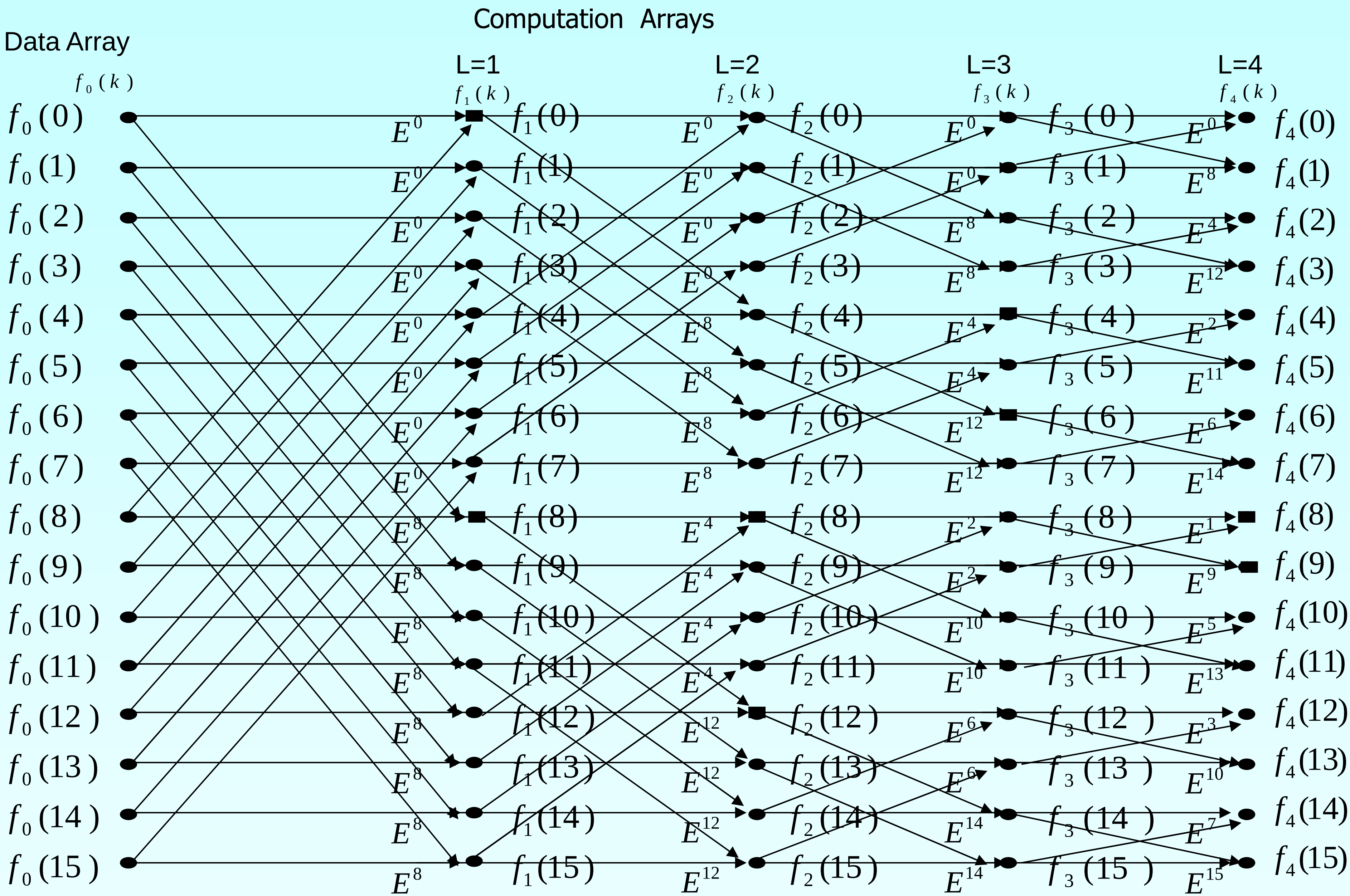

Figure 2. Graphical Form of FFT (Eq. 9) for the case

$$N = 2^r = 2^4 = 16$$

Companion Node Observation cont.

Furthermore, the computation of companion nodes are independent of other nodes (within the l^{th} –vector). Therefore, the computed $f_1(0)$ and $f_1(8)$ will override the original space of $f(0)$ and $f(8)$.

Similarly, the computed $f_2(8)$ and $f_2(12)$ will override the space occupied by $f_1(8)$ and $f_1(12)$ which in turn, will occupy the original space of $f(8)$ and $f(12)$.

Hence, only one complex vector (or 2 real vectors) of length N are needed for the entire FFT process.

Companion Node Spacing

Observing Figure 2, the following statements can be made:

a) in the first vector ($l = 1$), the companion $f_1(0)$ nodes $f_1(8)$ and are separated by $k = 8\left(or\ \frac{N}{2^l} = \frac{16}{2^1}\right)$

b) in the second vector ($l = 2$), the companion nodes $f_2(8)$ and $f_2(12)$ are separated by $k = 4$.

$(or\ \frac{N}{2^l} = \frac{16}{2^2} = \frac{16}{4}), etc.$

Companion Node Computation

The operation counts in any companion nodes (of the $l^{th} = 2^{nd}$ vector), such as $f_2(8)$ and $f_2(12)$ can be explained as (see Figure 2).

$$f_2(8) = f_1(8) + f_1(12) \times E^4 \quad (15)$$

$$\begin{aligned} f_2(12) &= f_1(8) + f_1(12) \times E^{12} \\ &= f_1(8) + f_1(12) \times E^8 E^4 \\ &= f_1(8) + f_1(12) \left[e^{-i\frac{2\pi}{(N=16)}} \right]^8 E^4 \\ &= f_1(8) + f_1(12) \left[e^{-i\pi} \right] E^4 \end{aligned}$$

$$f_2(12) = f_1(8) - f_1(12) \times E^4 \quad (16)$$

Companion Node Computation cont.

Thus, the companion nodes $f_2(8)$ and $f_2(12)$ computation will require 1 complex multiplication and 2 complex additions (see Eq. (15) and (16)). The weighting factors for the companion nodes ($f_2(8)$ and $f_2(12)$) are E^4(or E^U) and E^{12}(or $E^{U+N/2}$), respectively.

$$f_l(k) = f_{l-1}(k) + E^U f_{l-1}(k + \frac{N}{2^l}) \qquad (17)$$

$$f_l(k + \frac{N}{2^l}) = f_{l-1}(k) - E^U f_{l-1}(k + \frac{N}{2^l}) \qquad (18)$$

Skipping Computation of Certain Nodes

Because the pair of companion nodes k and $k+\frac{N}{2^L}$ are separated by the "distance" $\frac{N}{2^L}$, at the L^{th} level, after every $\frac{N}{2^L}$ node computation, then the next $\frac{N}{2^L}$ nodes will be skipped. (see Figure 2)

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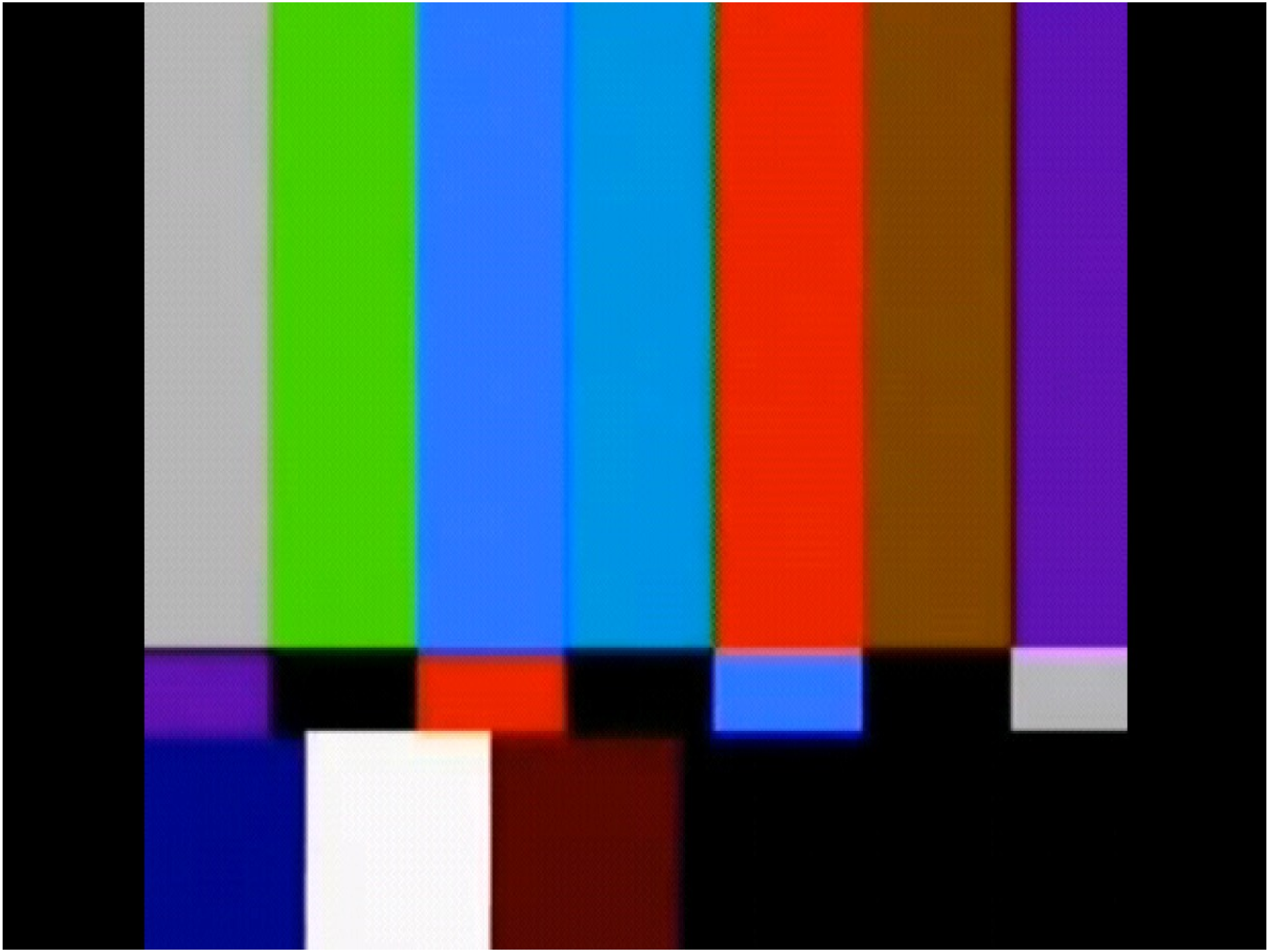

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Part: Determination of E^U

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Lecture # 14

Chapter 11.05: Determination of E^U

The values of "U"

$$f_l(k) = f_{l-1}(k) + E^U f_{l-1}\left(k + \frac{N}{2^l}\right);$$

$$f_l\left(k + \frac{N}{2^l}\right) = f_{l-1}(k) - E^U f_{l-1}\left(k + \frac{N}{2^l}\right)$$

can be determined by the following steps:

<u>Step 1:</u> Express the index $k(= 0,1,2,\ldots, N-1)$ in binary form, using r bits. For $k = 8, L = 2, r = 4,$ and $N = 2^r = 2^4 = 16,$ one obtains:

$$k = 8 = 1,0,0,0 = (1)2^{r-1=3} + (0)2^2 + (0)2^1 + (0)2^0$$

Determination of E^U cont.

<u>Step 2:</u> Sliding this binary number $r-L=4-2=2$ positions to the right, and fill in zeros, the results are

$$1,0,0,0 \rightarrow X, X, 1,0 \rightarrow 0,0,1,0$$

It is important to realize that the results of Step 2 (0,0,1,0) are equivalent to expressing an integer $M=\frac{k}{2^{r-L}}=\frac{8}{2^{4-2}}=2$ in binary format. In other words

$$M=2=(0,0,1,0)$$

Determination of E^U cont.

<u>Step 3:</u> Reverse the order of the bits, then (0,0,1,0) becomes (0,1,0,0) = U.
Thus,

$$U = (0)2^3 + (1)2^2 + (0)2^1 + (0)2^0 = 4$$

It is "NOT" really necessary to perform Step 3, since the results of Step 2 can be used to compute "U" as following

$$U = (0)2^0 + (0)2^1 + (1)2^2 + (0)2^3 = 4$$

Computer Implementation to find E^U

Based on the previous discussions (with the 3-step procedures), to find the value of "U", one only needs a procedure to express an integer $M = \frac{k}{2^{r-L}}$ in binary format, with r bits.

Assuming M (a base 10 number) can be expressed as (assuming $r = 4$ bits)

$$M = a_4 a_3 a_2 a_1 = J_1 \tag{19}$$

Computer Implementation cont.

Divide M by 2, $J_2 = J_1/2$ then multiply the truncated result by 2 $(JJ_2 = J_2 \times 2)$, and compute the difference between the original number and the new number.

Computer Implementation cont.

Compute the difference between the original number and the new number $(= M = J_1) \& \& JJ_2$:

$$IDIFF = J_1 - JJ_2 \left\{ = M - \left(\frac{M}{2} \right)_{Truncated} \times 2 \right\} \qquad (20)$$

If *IDIFF = 0, then the bit* $a_1 = 0$

If *IDIFF ≠ 0, then the bit* $a_1 = 1$

Computer Implementation cont.

Once the bit a_1 has been determined, the value of J_1 is set to J_2(or value of J_1 is reduced by a factor of 2; since the previous .

$$J_1 = M = a_4 a_3 a_2 a_1$$

$$J_1 = M = a_4(2^3) + a_3(2^2) + a_2(2^1) + a_1(2^0)$$

A similar process can be used to determine the value of process can be used to determine the next bit a_2 etc.

Example 1

For $k = 8, N = 16 = 2^r, r = 4$ bits and $L = 2$.
Find the value of U.

$$M = \frac{k}{2^{r-L}} = \frac{8}{2^{4-2}} = 2 = J_1$$

Determine the bit a_1 (Index $I = 1$)
Initialize $U = 0$

$$J_2 = \frac{J_1}{2} = \frac{2}{2} = 1$$

$$IDIFF = J_1 - (JJ_2 = J_2 \times 2) = 2 - (1)(2) = 0$$

Thus $a_1 = 0$

$$U = U \times 2 + IDIFF = 0 \times 2 + 0 = 0$$

Example 1 cont.

Determine the bit a_2 (Index $I = 2$)

$$J_1 = J_2 = 1$$

$$J_2 = \frac{J_1}{2} = \frac{1}{2} = 0$$

$$IDIFF = J_1 - (JJ_2 = J_2 \times 2) = 1 - (0 \times 2) = 1$$

Thus $a_2 = 1$

$$U = U \times 2 + IDIFF = 0 \times 2 + 1 = 1$$

Example 1 cont.

Determine the bit a_3 (Index $I = 3$)

$$J_1 = J_2 = 0$$

$$J_2 = \frac{J_1}{2} = \frac{0}{2} = 0$$

$$IDIFF = J_1 - (JJ_2 = J_2 \times 2) = 0 - (0)(2) = 0$$

Thus $a_3 = 0$

$$U = U \times 2 + IDIFF = 1 \times 2 + 0 = 2$$

Example 1 cont.

Determine the bit a_4 (Index $I = 4$)

$$J_1 = J_2 = 0$$

$$J_2 = \frac{J_1}{2} = \frac{1}{2} = 0$$

$$IDIFF = J_1 - (JJ_2 = J_2 \times 2) = 0 - (0)(2) = 0$$

Thus $a_4 = 0$

$$U = U \times 2 + IDIFF = 2 \times 2 + 0 = 4$$

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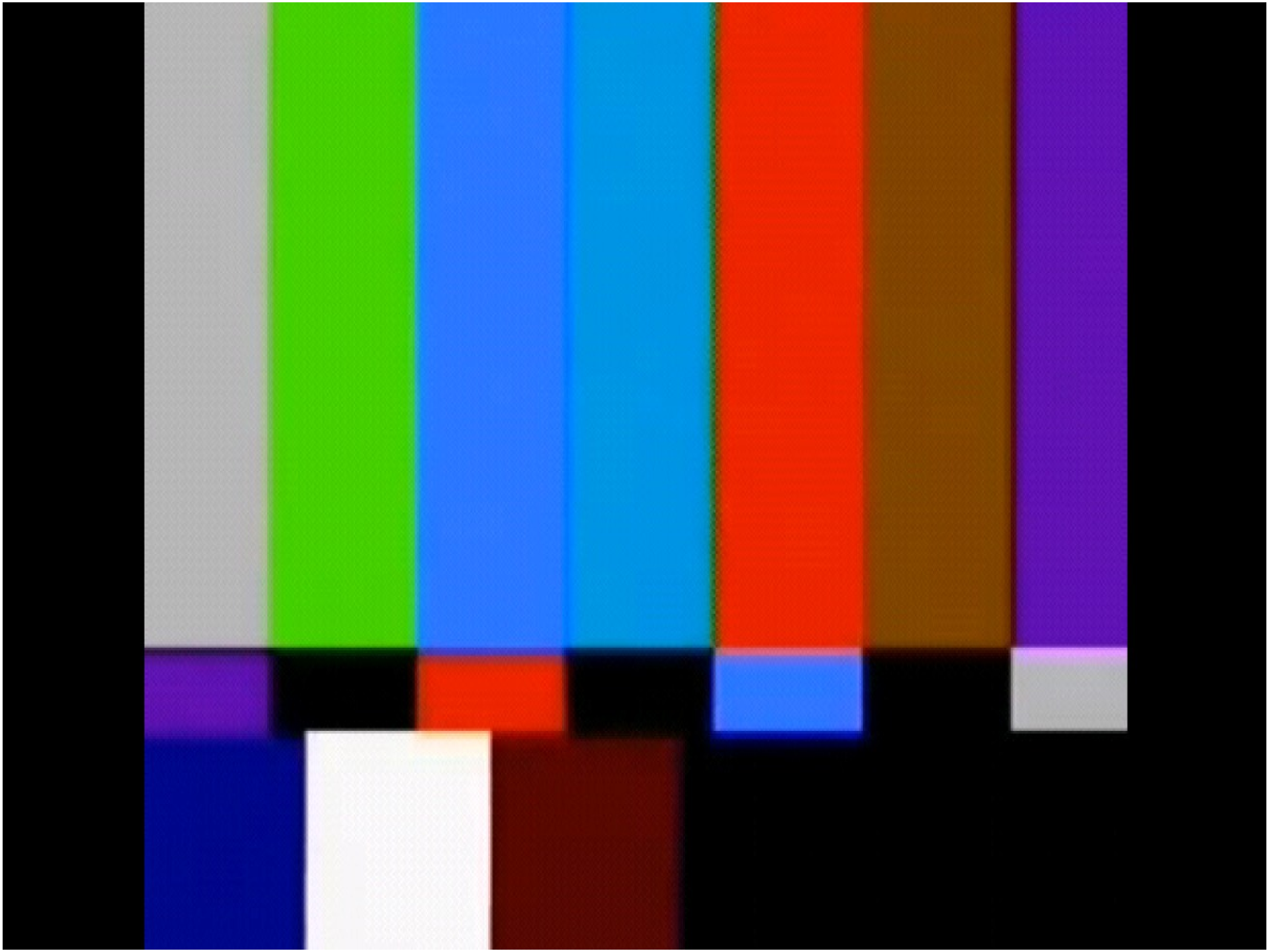

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Part: Unscrambling the FFT

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Lecture # 15

Chapter 11.05: Unscrambling the FFT (Contd.)

For the case

$$N = 16 = 2^{r=4}$$

, (see Figure 2), the final “bit-reversing” operation for FFT is shown in Figure 3.

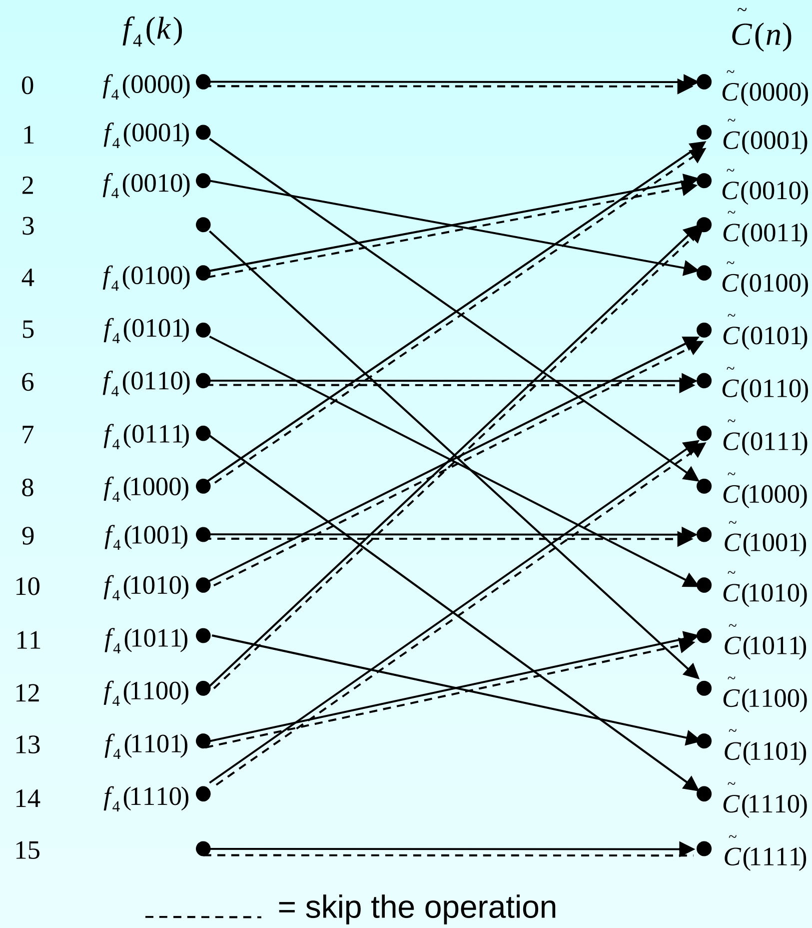

Figure 3. Final “bit-reversing” for FFT (with $N = 2^r = 2^4 = 16$)

For do-loop index $k = 0 = (0, 0, 0, 0) \Rightarrow i = (0, 0, 0, 0) =$ bit-reversion $= 0$

If (i.GT.k) Then

$T = f_4(k)$

$f_4(k) = f_4(i)$

$f_4(i) = T$

Endif

Hence, $f_4(0) = f_4(0)$ no swapping.

55

For $k = 1 = (0,0,0,1) \Rightarrow i = (1,0,0,0)$
$=$ bit-reversion $= 8$

If (i.GT.k) Then

$T = f_4(k=1)$

$f_4(k=1) = f_4(i=8)$

$f_4(i=8) = T$

Endif

Hence, $f_4(1) = f_4(8)$ are swapped.

.For $k=2=(0,0,1,0) \Rightarrow i = (0,1,0,0) = 4$
Hence, $f_4(2) = f_4(4)$; are swapped.

.For $k=3=(0,0,1,1) \Rightarrow i = (1,1,0,0) = 12$
Hence, $f_4(3) = f_4(12)$; are swapped.

. For $k=4=(0,1,0,0) \Rightarrow i=(0,0,1,0)=2$

In this case, since "i" is not greater than "k".
Hence, no swapping, since f_4 $(k = 2)$ and f_4 $(i = 4)$; had already been swapped earlier!
.etc.

Computer Implementation of FFT case for $N=2^r$

The pair of companion nodes computation are given by Eqs.(17) and (18). To avoid "complex number" operations,Eq.(17) can be computed based on "real number" operations, as following

$$\left\{f_L^R(k)+if_L^I(k)\right\}=\left\{f_{L-1}^R(k)+if_{L-1}^I(k)\right\}$$
$$+\left\{E^{U,R}+iE^{U,I}\right\}\times\left\{f_{L-1}^R\left(k+\frac{N}{2^L}\right)+if_{L-1}^I\left(k+\frac{N}{2^L}\right)\right\} \tag{21}$$

In Eq. (21), the superscripts R and I denote real and imaginary components, respectively.

Computer Implementation cont.

Multiplying the last 2 complex numbers, one obtains

$$\left\{ f_L^R(k) + i f_L^I(k) \right\} = \left\{ f_{L-1}^R(k) + i f_{L-1}^I(k) \right\}$$

$$+ \left\{ E^{U,R} \times f_{L-1}^R(k + \frac{N}{2^L}) - E^{U,I} \times f_{L-1}^I(k + \frac{N}{2^L}) \right\}$$

$$+ i \left\{ E^{U,R} \times f_{L-1}^I(k + \frac{N}{2^L}) + E^{U,I} \times f_{L-1}^R(k + \frac{N}{2^L}) \right\} \quad (22)$$

Equating the real (and then, imaginary) components on the Left-Hand-Side (LHS), and the Right-Hand-Side (RHS) of Eq. (22), one obtains

Computer implementation cont.

$$\left\{ f_L^R(k) \right\} = \left\{ f_{L-1}^R(k) \right\} + \left\{ E^{U,R} \times f_{L-1}^R(k + \frac{N}{2^L}) - E^{U,I} \times f_{L-1}^I(k + \frac{N}{2^L}) \right\}$$

(23A)

$$\left\{ f_L^I(k) \right\} = \left\{ f_{L-1}^I(k) \right\} + \left\{ E^{U,R} \times f_{L-1}^I(k + \frac{N}{2^L}) + E^{U,I} \times f_{L-1}^R(k + \frac{N}{2^L}) \right\}$$

(23B)

Computer implementation cont.

Recall Eq. (4)

$$E = e^{-i\frac{2\pi}{N}}$$

Hence

$$E^U = \left(e^{-i\frac{2\pi}{N}} \right)^U = e^{-i\frac{2\pi U}{N}} = e^{-i\theta} = \cos(\theta) - i\sin(\theta) \qquad (24)$$

where

$$\theta = \frac{2\pi U}{N} = \frac{6.28U}{N} \qquad (25)$$

Thus:

$$E^{U,R} = \cos(\theta) \qquad (26A)$$

$$E^{U,I} = -\sin(\theta) \qquad (26B)$$

Computer Implementation cont.

Substituting Eqs. (26A) and (26B) into Eqs. (23A) and (23B), one gets

$$\left\{f_L^R(k)\right\} = \left\{f_{L-1}^R(k)\right\} + \left\{\cos(\theta) \times f_{L-1}^R\left(k + \frac{N}{2^L}\right) + \sin(\theta) \times f_{L-1}^I\left(k + \frac{N}{2^L}\right)\right\} \quad \text{(27A)}$$

$$\left\{f_L^I(k)\right\} = \left\{f_{L-1}^I(k)\right\} + \left\{\cos(\theta) \times f_{L-1}^I\left(k + \frac{N}{2^L}\right) - \sin(\theta) \times f_{L-1}^R\left(k + \frac{N}{2^L}\right)\right\} \quad \text{(27B)}$$

Similarly, the single (complex number) Eq. (18) can be expressed as 2 equivalent (real number) Eqs. Like Eqs. (27A) and (27B).

THE END

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Acknowledgement

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This material is based upon work supported by the National Science Foundation under Grant # 0717624. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.