Chapter 11.03
Fourier Transform Pair: Frequency and Time Domain

Introduction

In Chapter 11.02, Fourier approximations were expressed in the time domain. The amplitude (vertical axis) of a given periodic function can be plotted versus time (horizontal axis), but it can also be plotted in the frequency domain [1-6] as shown in Figure 1.

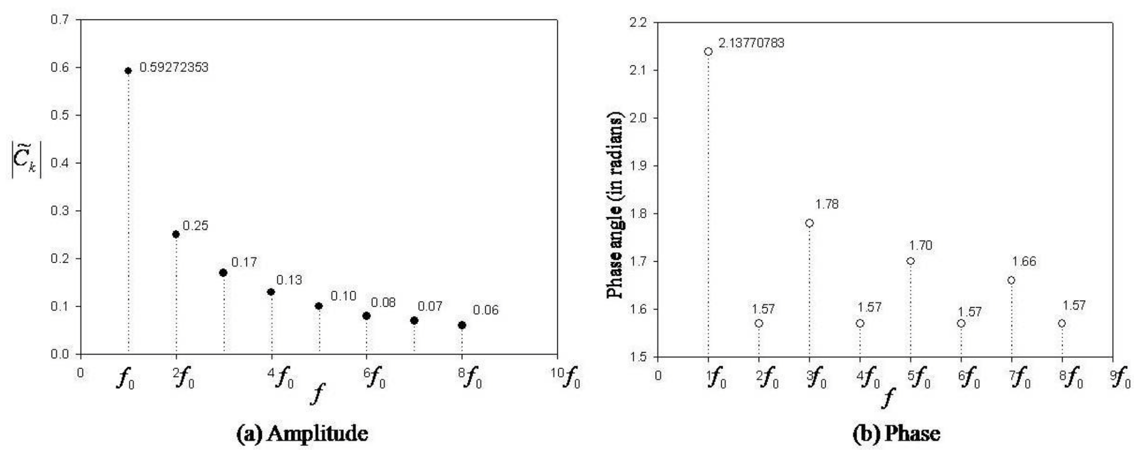

Figure 1 Periodic Function (see Example 1 in Chapter 11.02) In Frequency Domain.

The advantages of plotting the amplitude of a given periodic function in frequency domain (instead of time domain) are due to the following reasons:

For a specific value "k" (say $k = 2$) of the Fourier series in the time domain, one has to plot the entire curve to observe the amplitude of a given periodic function (recall $\bar{f}_2(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t)$, see Example 1 in Chapter 11.02). However, in the frequency domain, the amplitude can be plotted as a single point. (see Figure 1a).

In the frequency domain, one can easily identify which frequency (or corresponding to which value of "k") contributes the most to the amplitude [see Figure 1(a)], where such information is not readily available if time domain is used.

From the amplitude plot in frequency domain [see Figure 1(a)], one can easily identify that contributions to the amplitude beyond the 8th frequency ($k > 8$) are not significant any more.

In real-life structural dynamics problems, such as the dynamical (time-dependent) response of a (building) structure subjected to oscillated loads (for example, the operational machines attached to the structures), the displacement superposition method is often used to predict the (time dependent) displacement response of the structure. This method basically transforms the original (large, coupled) equation of motion into a reduced (much smaller size, un-coupled) equation of motion by making use of the few free vibration mode shapes and its associated frequencies. Knowledge of which frequencies (and the corresponding mode shapes) that have the most contribution to the predicted dynamical response (such as nodal displacement response) plays crucial roles for the algorithms' efficiencies.

Detailed explanations on how to obtain Figures 1(a), and 1(b) are now presented in the following sections.

Explanation of Figure 1(a) and 1(b)

One starts with Equation (18) and (20) of Chapter 11.02

$$f(t) = \sum_{k=-\infty}^{\infty} \widetilde{C}_k e^{ikw_0 t}$$

where

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t}\, dt\right\}$$

For the periodic function shown in Example 1 of Chapter 11.02 (or Figure 1 of Chapter 11.02), one has

$$\begin{aligned} w_0 &= 2\pi f \\ &= \frac{2\pi}{T} \\ &= \frac{2\pi}{2\pi} \\ &= 1 \end{aligned}$$

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^{\pi} t \times e^{-ikt}\, dt + \int_{\pi}^{2\pi} \pi \times e^{-ikt}\, dt\right\}$$

Define, and using "integration by parts" formula

$$A \equiv \int_0^{\pi} t \times e^{-ikt}\, dt = \left[t \times \left(\frac{-1}{ik}\right) e^{-ikt}\right]_0^{\pi} + \int_0^{\pi} \left(\frac{1}{ik}\right) e^{-ikt}\, dt$$

$$\begin{aligned} A &= \left[\left(\frac{-\pi}{ik}\right) e^{-ik\pi}\right]_0^{\pi} + \left(\frac{1}{ik}\right)\left[\left(-\frac{1}{ik}\right) e^{-ikt}\right]_0^{\pi} \\ &= \left[\left(\frac{-\pi}{ik}\right) e^{-ik\pi}\right] + \left(\frac{1}{k^2}\right)\left[e^{-ik\pi} - 1\right] \end{aligned}$$

$$= \left[\left(\left(\frac{\pi i}{k} \right) e^{-ik\pi} + \left(\frac{1}{k^2} \right) e^{-ik\pi} - \frac{1}{k^2} \right) \right]$$

$$B \equiv \pi \int_{\pi}^{2\pi} e^{-ikt} dt = \left[\left(e^{-ikt} \right) \left(\frac{-\pi}{ik} \right) \right]_{\pi}^{2\pi}$$

$$= \left(\frac{-\pi}{ik} \right) \left[e^{-ik2\pi} - e^{-ik\pi} \right]$$

$$= \left(\frac{\pi i}{k} \right) \left[e^{-ik2\pi} - e^{-ik\pi} \right]$$

Thus,

$$\widetilde{C}_k = \left(\frac{1}{2\pi} \right) \{ A + B \}$$

$$= \left(\frac{1}{2\pi} \right) \left\{ e^{-ik\pi} \left(\frac{\pi i}{k} + \frac{1}{k^2} - \frac{\pi i}{k} \right) - \frac{1}{k^2} + \left(\frac{\pi i}{k} \right) e^{-ik2\pi} \right\}$$

Using the following Euler identities

$$e^{-ik\pi} = \cos(-k\pi) + i\sin(-k\pi)$$

$$= \cos(k\pi) - i\sin(k\pi)$$

$$= \cos(k\pi)$$

$$e^{-ik(2\pi)} = \cos(k(2\pi)) - i\sin(k(2\pi))$$

$$= \cos(k(2\pi))$$

Hence, one obtains (noting that $\cos(k2\pi) = 1$, for any integer k):

$$\widetilde{C}_k = \left(\frac{1}{2\pi} \right) \left\{ \cos(k\pi) \times \left(\frac{1}{k^2} \right) - \frac{1}{k^2} + \left(\frac{\pi i}{k} \right) \cos(k2\pi) \right\}$$

or,

$$\widetilde{C}_k = \left(\frac{1}{2\pi} \right) \left\{ \left(\frac{1}{k^2} \right) \cos(k\pi) - \frac{1}{k^2} + \left(\frac{\pi i}{k} \right) \right\}$$

Also, since:

$$\cos(k\pi) = \begin{cases} -1 \text{ } for \text{ } k = \text{odd integer } (=1,3,5,7,...) \\ +1 \text{ } for \text{ } k = \text{even integer } (=2,4,6,8,...) \end{cases}$$

Hence:

$$\cos(k\pi) = (-1)^k$$

Thus,

$$\widetilde{C}_k = \left(\frac{1}{2\pi} \right) \left\{ \frac{(-1)^k}{k^2} - \frac{1}{k^2} + \frac{\pi i}{k} \right\}$$

$$\widetilde{C}_k = \left(\frac{1}{2\pi k^2} \right) \left[(-1)^k - 1 \right] + \left(\frac{1}{2k} \right) i$$

From Equation (15) in Chapter 11.02, one has:

$$\widetilde{C}_k = \frac{a_k - ib_k}{2}$$

Hence upon comparing the above 2 equations, one concludes

$$a_k \equiv \left(\frac{1}{\pi k^2}\right)\left[(-1)^k - 1\right]$$

$$b_k = \left(\frac{-1}{k}\right)$$

Remarks:

For $k = 1,2,3,4,...,8;$ the values for a_k and b_k (based on the above 2 formulas) are exactly identical as the ones presented earlier in Example 1 in Chapter 11.02.

Thus

$$\begin{aligned}\tilde{C}_1 &= \frac{a_1 - ib_1}{2} \\ &= \frac{\frac{-2}{\pi} - i(-1)}{2} \\ &= \frac{-1}{\pi} + \frac{1}{2}i\end{aligned}$$

$$\begin{aligned}\tilde{C}_2 &= \frac{a_2 - ib_2}{2} \\ &= \frac{0 - i\left(-\frac{1}{2}\right)}{2} \\ &= 0 + \frac{1}{4}i\end{aligned}$$

$$\begin{aligned}\tilde{C}_3 &= \frac{a_3 - ib_3}{2} \\ &= \frac{\left(\frac{-2}{9\pi}\right) - i\left(\frac{-1}{3}\right)}{2} \\ &= \left(\frac{-1}{9\pi}\right) + \frac{1}{6}i\end{aligned}$$

$$\begin{aligned}\tilde{C}_4 &= \frac{a_4 - ib_4}{2} \\ &= \frac{0 - i\left(\frac{-1}{4}\right)}{2} \\ &= 0 + \frac{1}{8}i\end{aligned}$$

$$\tilde{C}_5 = \frac{a_5 - ib_5}{2}$$

$$= \frac{\left(\frac{-2}{25\pi}\right) - i\left(\frac{-1}{5}\right)}{2}$$

$$= \left(\frac{-1}{25\pi}\right) + \frac{1}{10} i$$

$$\tilde{C}_6 = \frac{a_6 - ib_6}{2}$$

$$= \frac{0 - i\left(\frac{-1}{6}\right)}{2}$$

$$= 0 + \frac{1}{12} i$$

$$\tilde{C}_7 = \frac{a_7 - ib_7}{2}$$

$$= \frac{\left(\frac{-2}{49\pi}\right) - i\left(\frac{-1}{7}\right)}{2}$$

$$= \left(\frac{-1}{49\pi}\right) + \frac{1}{14} i$$

$$\tilde{C}_8 = \frac{a_8 - ib_8}{2}$$

$$= \frac{0 - i\left(\frac{-1}{8}\right)}{2}$$

$$= 0 + \frac{1}{16} i$$

In general, one has

$$\tilde{C}_k = \begin{cases} \frac{-1}{k^2\pi} + \left(\frac{1}{2k}\right) i \ \ for\ k = 1,3,5,7,.. = \text{odd integer} \\ \left(\frac{1}{2k}\right) i \ \ for\ k = 2,4,6,8,.. = \text{even integer} \end{cases}$$

Representation of a complex number in polar coordinates

In Cartesian (rectangular) coordinates, a complex number $\widetilde{C}_k$ can be expressed as:

$$\widetilde{C}_k = R_k + (I_k)i$$

where R_k and I_k represents the real and imaginary components of $\widetilde{C}_k$, respectively.
In polar coordinates, a complex number $\widetilde{C}_k$ can be expressed as:

$$\widetilde{C}_k = Ae^{i\theta} = A\{\cos(\theta) + i\sin(\theta)\} = \{A\cos(\theta)\} + \{A\sin(\theta)\}i$$

where A and θ represents the amplitude and phase angle of $\widetilde{C}_k$, respectively (see Figure 2).

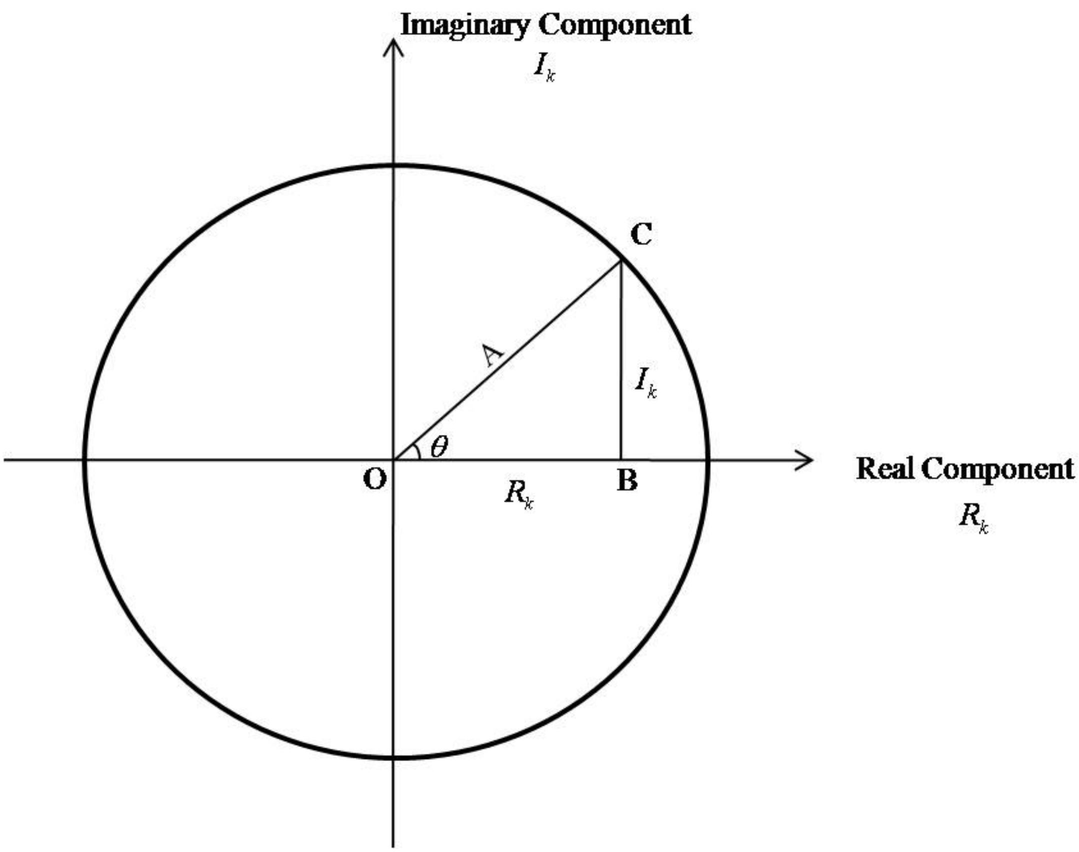

Figure 2 Representation of a complex number in polar coordinates

Thus, one obtains the following relations between the cartesian and polar coordinate systems:

$$R_k = A\cos(\theta)$$

$$I_k = A\sin(\theta)$$

Hence:

$$R_k^2 + I_k^2 = A^2\cos^2(\theta) + A^2\sin^2(\theta) = A^2\left[\cos^2(\theta) + \sin^2(\theta)\right]$$

$$A^2 = R_k^2 + I_k^2$$

$$A = \sqrt{R_k^2 + I_k^2}$$

$$\cos(\theta) = \frac{R_k}{A} \text{ implies } \theta = \cos^{-1}\left(\frac{R_k}{A}\right)$$

$$\sin(\theta)=\frac{I_k}{A} \text{ implies } \theta=\sin^{-1}\left(\frac{I_k}{A}\right)$$

Based on the above 3 formulas, the complex numbers $\widetilde{C}_k$, for $k=1,2,3,...,8$ can be expressed as

$$\widetilde{C}_1=\frac{-1}{\pi}+\left(\frac{1}{2}\right)i$$
$$=(0.59272353)e^{i(2.13770783)}$$

Hence, the amplitude A and Phase angle θ for $\widetilde{C}_1$ are 0.59272353, and 2.13770783 radians, respectively. The readers should refer to Figures 1(a) and 1(b) to confirm the plotted values.

Important Notes

If one uses the formula

$$\theta=\cos^{-1}\left(\frac{R_k}{A}\right)$$
$$=\cos^{-1}\left(\frac{\frac{-1}{\pi}}{0.59272353}\right)$$
$$=2.13770783 \text{ radians}$$
$$=122.48^\circ$$

However, the other formula for θ gives:

$$\theta=\sin^{-1}\left(\frac{I_k}{A}\right)$$
$$=\sin^{-1}\left(\frac{0.5}{0.59272353}\right)$$
$$=1.0038848 \text{ radians}$$
$$=57.52^\circ$$

Since R_k is negative, and I_k is positive, the angle θ must be in the 2nd (or upper left) quadrant of a circle (or $90^\circ \le \theta \le 180^\circ$). Thus, the correct value for θ should be 2.13770783 radians (or 122.48^o) and the other value for $\theta=1.0038848$ radians must be discarded.

Similarly, one obtains

$$\widetilde{C}_2=0+\frac{1}{4}i$$
$$=(0.25)e^{i\left(\frac{\pi}{2}\right)}$$
$$=(0.25)e^{i(1.57079633)}$$

$$\widetilde{C}_3 = \left(\frac{-1}{9\pi}\right) + \frac{1}{6}i$$
$$= (0.17037798)e^{i(1.77990097)}$$
$$\widetilde{C}_4 = 0 + \frac{1}{8}i$$
$$= (0.125)e^{i\left(\frac{\pi}{2}\right)}$$
$$= (0.125)e^{i(1.57079633)}$$
$$\widetilde{C}_5 = \left(\frac{-1}{25\pi}\right) + \frac{1}{10}i$$
$$= (0.100807311)e^{i(1.69743886)}$$
$$\widetilde{C}_6 = 0 + \frac{1}{12}i$$
$$= (0.08333333)e^{i\left(\frac{\pi}{2}\right)}$$
$$= (0.08333333)e^{i(1.57079633)}$$
$$\widetilde{C}_7 = \left(\frac{-1}{49\pi}\right) + \frac{1}{14}i$$
$$= (0.07172336)e^{i(1.66149251)}$$
$$\widetilde{C}_8 = 0 + \frac{1}{16}i$$
$$= (0.0625)e^{i\left(\frac{\pi}{2}\right)}$$

In summary, the given periodic function (shown in Example 1 of Chapter 11.02) can also be expressed in complex number formats, in polar coordinate with the amplitudes and phase angles given in the following table (also refer to Figures 1(a), and 1(b)).

Table 1 Amplitude and phase angle (in radians) for varying k values.

k	*Amplitude*	*Phase Angle* (*radians*)
1	0.59272353	2.13770783
2	0.25	$\frac{\pi}{2} = 1.57079633$
3	0.14037798	1.77990097
4	0.125	$\frac{\pi}{2}$
5	0.100807311	1.69743886
6	0.08333333	$\frac{\pi}{2}$
7	0.07172336	1.66149251
8	0.0625	$\frac{\pi}{2}$

Non-Periodic Function

Recall that a periodic function can be expressed in terms of the exponential form, accordingly to Equations (18, 20) of Chapter 11.02 as

$$f(t) = \sum_{k=-\infty}^{\infty} \widetilde{C}_k e^{ikw_0 t}$$

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\}$$

Define the following function

$$\hat{F}(ikw_0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ikw_0 t} dt \tag{1}$$

where $\hat{F}(ikw_0)$ is a function of $i, k,$ and w_0

Then, Equation (20) of Chapter 11.02 can be written as

$$\widetilde{C}_k = \left(\frac{1}{T}\right) \times \hat{F}(ikw_0) \tag{2}$$

and Equation (18) of Chapter 11.02 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T}\right) \times \hat{F}(ikw_0) e^{ikw_0 t} \tag{3}$$

A non-periodic function f_{np} can be considered as a periodic function, with the period

$$T \to \infty, \text{ or } \Delta f \equiv \frac{1}{T} \to 0 \text{ (see Figure 3)}$$

From Equations (6) and (7) from Chapter 11.01, one gets

$$\begin{aligned} w_0 &= 2\pi f \\ &= \frac{2\pi}{T} \\ &= 2\pi(\Delta f) \end{aligned} \tag{4}$$

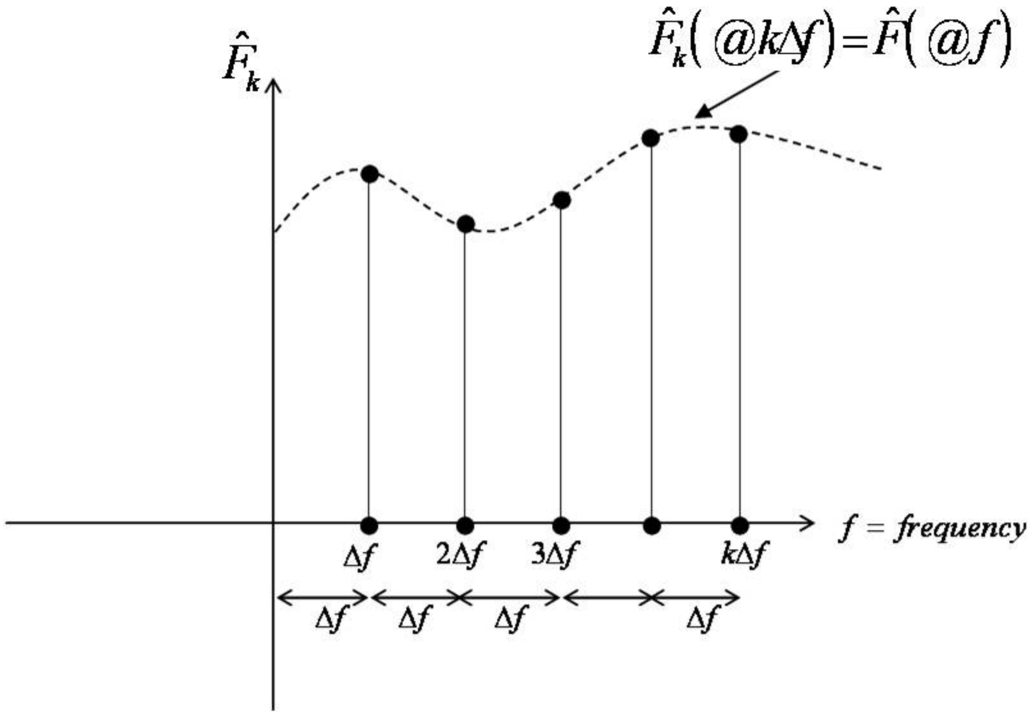

Figure 3 Discretization of frequency data.

From Equation (3), one obtains

$$f_{np}(t) = \lim_{\substack{T\to\infty \\ or \Delta f\to 0}} f(t)$$

$$= \lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} (\Delta f) \times \hat{F}(ikw_0)e^{ikw_0 t} \tag{5}$$

In the above equation, the subscript "np" denotes non-periodic function.

$$f_{np}(t) = \lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} (\Delta f) \times \hat{F}(ik2\pi\Delta f)e^{ik2\pi\Delta f t} \tag{6}$$

Realizing that $k\Delta f = f$ (See Figure 3), the above equation becomes

$$f_{np}(t) = \int df \times \hat{F}(i2\pi f)e^{i2\pi f t}$$

$$f_{np}(t) = \int \hat{F}(i2\pi f)e^{i2\pi f t} df \tag{7}$$

Multiplying and dividing the right-hand-side of the equation by 2π, one obtains

$$f_{np}(t) = \left(\frac{1}{2\pi}\right)\int \hat{F}(i2\pi f)e^{i2\pi f t} d(2\pi f)$$

$$= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \hat{F}(iw_0)e^{iw_0 t} d(w_0) \text{; inverse Fourier transform} \tag{8}$$

Using the definition stated in Equation (1), one has

$$\hat{F}(iw_0) = \int_{-\infty}^{\infty} f_{np}(t)e^{-iw_0 t} d(t) \text{; Fourier transform} \tag{9}$$

Thus, Equations (9) and (8) will transform a non-periodic function from time domain to frequency domain, and from frequency domain to time domain, respectively.

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FAST FOURIER TRANSFORM	
Topic	Fourier Transform Pair: Frequency and Time Domain
Summary	Textbook notes on Fourier Transform Pair: Frequency and Time Domain
Major	General Engineering
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