

Chapter 11.04

Discrete Fourier Transform

Introduction

Recalled the exponential form of Fourier series (see Equations 18 and 20 from Chapter 11.02),

$$f(t) = \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{ikw_0 t} \quad (18, \text{Ch. 11.02})$$

$$\tilde{C}_k = \left(\frac{1}{T} \right) \left\{ \int_0^T f(t) \times e^{-ikw_0 t} dt \right\} \quad (20, \text{Ch. 11.02})$$

While the above integral can be used to compute \tilde{C}_k , it is more preferable to have a discretized formula version to compute \tilde{C}_k . Furthermore, the Discrete Fourier Transform (or DFT) [1–5] will also facilitate the development of much more efficient algorithms for Fast Fourier Transform (or FFT), to be discussed in Chapters 11.05 and 11.06.

Derivations of DFT Formulas

If time “ t ” is discretized at $t_1 = \Delta t, t_2 = 2\Delta t, t_3 = 3\Delta t, \dots, t_n = n\Delta t$,

Then Equation (18, of Chapter 11.02) becomes

$$f(t_n) = \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 t_n} \quad (1)$$

To simplify the notation, define

$$t_n = n \quad (2)$$

Then, Equations (1) can be written as

$$f(n) = \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 n} \quad (3)$$

In the above formula, “ n ” is an integer counter. However, $f(n)$ and t_n do NOT have to be integer numbers.

Multiplying both sides of Equation (3) by $e^{-ilw_0 n}$, and performing the summation on “ n ”, one obtains (note: $l = \text{integer number}$)

$$\sum_{n=0}^{N-1} f(n) \times e^{-ilw_0 n} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 n} \times e^{-ilw_0 n} \quad (4)$$

$$= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{C}_k e^{i(k-l)w_0 n} \quad (5)$$

$$= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{C}_k e^{i(k-l)\frac{2\pi}{N}n} \quad (6)$$

Switching the order of summations on the right-hand-side of Equation (6), one obtains

$$\sum_{n=0}^{N-1} f(n) \times e^{-il\left(\frac{2\pi}{N}\right)n} = \sum_{k=0}^{N-1} \tilde{C}_k \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \quad (7)$$

Define

$$A = \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \quad (8)$$

There are 2 possibilities for $(k-l)$ to be considered in Equation (8)

Case(1): $(k-l)$ is a multiple integer of N , such as

$$(k-l) = mN; \text{ or } k = l + mN \text{ where } m = 0, \pm 1, \pm 2, \dots$$

Thus, Equation (8) becomes:

$$\begin{aligned} A &= \sum_{n=0}^{N-1} e^{im2\pi n} \\ &= \sum_{n=0}^{N-1} \cos(mn2\pi) + i \sin(mn2\pi) \end{aligned} \quad (9)$$

Hence:

$$A = N \quad (10)$$

Case(2): $(k-l)$ is NOT a multiple integer of N

In this case, from Equation (8) one has

$$A = \sum_{n=0}^{N-1} \left\{ e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \right\} \quad (11)$$

Define:

$$\begin{aligned} a &= e^{i(k-l)\frac{2\pi}{N}} \\ &= \cos\left\{(k-l)\frac{2\pi}{N}\right\} + i \sin\left\{(k-l)\frac{2\pi}{N}\right\} \end{aligned} \quad (12)$$

$$a \neq 1; \text{ because } (k-l) \text{ is "NOT" a multiple integer of } N \quad (13)$$

Then, Equation (11) can be expressed as

$$A = \sum_{n=0}^{N-1} a^n \quad (14)$$

From mathematical handbooks, the right side of Equation (14) represents the “geometric series”, and can be expressed as

$$A = \sum_{n=0}^{N-1} a^n = N \text{ if } a = 1 \quad (15)$$

$$= \frac{1 - a^N}{1 - a} \text{ if } a \neq 1 \quad (16)$$

Because of Equation (13), hence Equation (16) should be used to compute A . Thus

$$A = \frac{1 - a^N}{1 - a} \text{ (See Equation (12))} \quad (17)$$

$$= \frac{1 - e^{i(k-l)2\pi}}{1 - a}$$

Since $(k - l)$ is still a multiple of 2π , hence

$$e^{i(k-l)2\pi} \equiv \cos\{(k-l)2\pi\} + i \sin\{(k-l)2\pi\} \quad (18)$$

$$= 1$$

Substituting Equation (17) into Equation (18), one gets

$$A = 0 \quad (19)$$

Thus, combining the results of case (1) and case (2), one gets (see Equations (10) and Equation (19))

$$A = N + 0 \quad (20)$$

Substituting Equation (20) into Equation (8), and then referring to Equation (7), one gets

$$\sum_{n=0}^{N-1} f(n)e^{-ilw_0n} = \sum_{k=0}^{N-1} \tilde{C}_k \times N \quad (20a)$$

Recalled $k = l + mN$ (where l, m are integer numbers), and since k must be in the range $0 \rightarrow N - 1$, therefore $m = 0$. Thus:

$$k = l + mN \text{ becomes } k = l$$

Equation (20a) can, therefore, be simplified to

$$\sum_{n=0}^{N-1} f(n)e^{-ilw_0n} = \tilde{C}_l \times N \quad (20b)$$

Thus

$$\tilde{C}_l = \tilde{C}_k = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} f(n)e^{-ikw_0n} \quad (21)$$

$$= \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} f(n) \{\cos(kw_0n) - i \sin(kw_0n)\}$$

where

$$n \equiv t_n$$

and

$$f(n) = \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0n} \quad (3, \text{ repeated})$$

$$= \sum_{k=0}^{N-1} \tilde{C}_k \{\cos(kw_0n) + i \sin(kw_0n)\}$$

Remarks:

(a) Consider the exponential term in Equation (1). Let

$$E = e^{(ikw_0n)} = e^{(ik \times \frac{2\pi}{N} * n)}$$

If one replaces “ n ” by “ $-(N - n)$ ” (or “ $(n - N)$ ”) into the above equation, then one obtains

$$\begin{aligned} e^{ik \times \frac{2\pi}{N} \times (n-N)} &= e^{(ik \times \frac{2\pi}{N} * n)} \times [e^{(-ik \times 2\pi)} = 1] \\ &= E \end{aligned}$$

Thus, Equation (1) indicates that the force corresponding to frequencies of order “ n ” and “ $-(N - n) = n - N$ ” have the same values. Hence

$$\begin{aligned} w_n &= n\bar{w} \quad \text{for } n \leq \frac{N}{2} \\ &= -(N - n)\bar{w} \quad \text{for } n > \frac{N}{2} \end{aligned}$$

and the frequency corresponding to $n = \frac{N}{2}$ is the highest frequency that can be considered in the discrete Fourier series ($w_{\frac{N}{2}}$ is called the Nyquist frequency). If there are harmonic (force) components above $w_{\frac{N}{2}}$ in the original function, then these higher components will introduce

distortions in the lower harmonic components (known as ALIASING phenomenon). Because of the ALIASING phenomenon, the number of (N) data points should be “at least twice” the highest harmonic component presents in the (forcing) function, for sufficient computational accuracy. As an example, if the forcing function is given as

$$F(t) = \sum_{n=1}^{16} 100 \times \cos(2\pi nt)$$

then, the minimum value of N (= Number of sample data points) should be $N_{\min} = 32$.

(b) The factor $\left(\frac{1}{N}\right)$, shown in the DFT Equation (21), is merely a scale factor. It can also be placed in the inverse Fourier Transform Equation (1), but not both.

Thus, Equations (21) and (1) can be re written as

$$\tilde{C}_n = \sum_{k=0}^{N-1} f(k) e^{-ik \left(w_0 = \frac{2\pi}{N} \right) n} \quad (22)$$

$$f(k) = \left(\frac{1}{N} \right) \sum_{n=0}^{N-1} \tilde{C}_n e^{ik \left(w_0 = \frac{2\pi}{N} \right) n} \quad (23)$$

To avoid computation with “complex numbers”, Equation (22) can be expressed as

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1} \left\{ f^R(k) + i f^I(k) \right\} \times \left\{ \cos(\theta) - i \sin(\theta) \right\} \quad (22a)$$

where

$$\theta = k \left(w_0 = \frac{2\pi}{N} \right) n \quad (22b)$$

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1} \left\{ f^R(k) \times \cos(\theta) + f^I(k) \sin(\theta) \right\} + i \left\{ f^I(k) \cos(\theta) - f^R(k) \sin(\theta) \right\}$$

The above “complex number” equation is equivalent to the following 2 “real number” equations

$$\tilde{C}_n^R = \sum_{k=0}^{N-1} \{f^R(k) \cos(\theta) + f^I(k) \sin(\theta)\} \quad (22c)$$

$$\tilde{C}_n^I = \sum_{k=0}^{N-1} \{f^I(k) \cos(\theta) - f^R(k) \sin(\theta)\} \quad (22d)$$

Computer program implementation for the DFT equations (22c, 22d) are given at <http://numericalmethods.eng.usf.edu/simulations/mtl/11fft/dft.m>.

Detailed Explanation About Aliasing Phenomenon, Nyquist Samples, Nyquist Rate.

When a function $f(t)$, which may represent the signals from some real-life phenomenon (shown in Figure 1), is sampled, it basically converts that function into a sequence $\tilde{f}(k)$ at discrete locations of t . These discrete locations are assumed to have “equally spaced and the distance between any 2 samples is Δt ”. Thus, $\tilde{f}(k)$ represents the value of $f(t)$, at $t = t_0 + k\Delta t$, where t_0 is the location of the first sample (at $k = 0$). If the sample locations were done properly, then the original function $f(t)$, can be recovered through interpolation process of these discrete sample values.

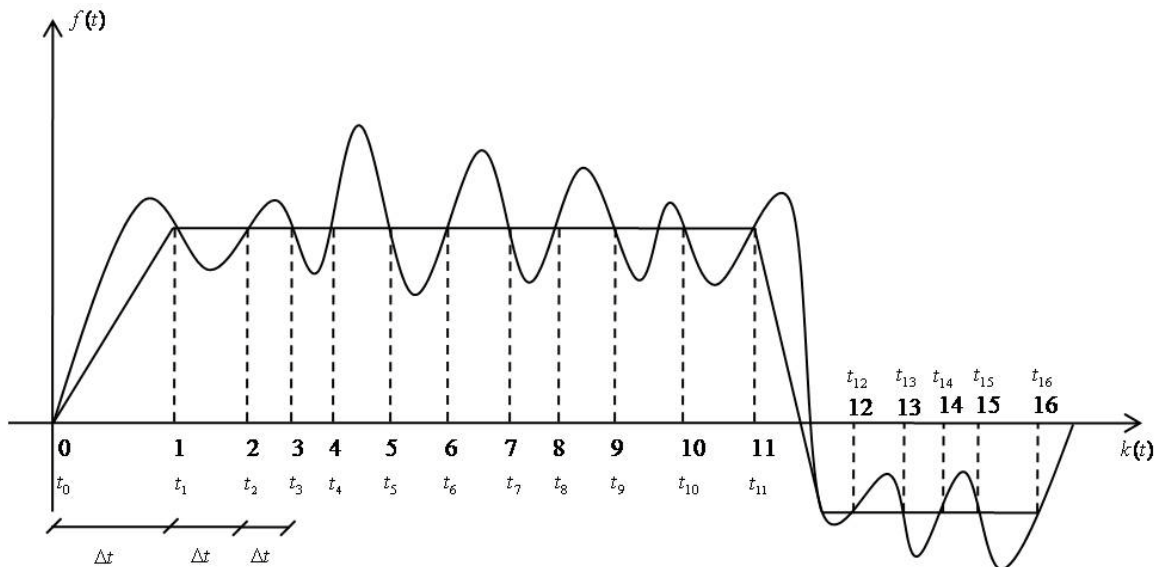


Figure 1 Function to be Sampled and “Aliased” Sample Problem.

In Figure 1, the samples have been taken with a fairly large Δt . Thus, these sequence of discrete data will not be able to recover the original signal function $f(t)$. For example, if all discrete values of $f(t)$, were connected by piecewise linear fashion, then a nearly horizontal straight line will occur between t_1 through t_{11} , and t_{12} through t_{16} , respectively (See Figure 1). These piecewise linear interpolation (or other interpolation schemes will NOT produce a curve which resemble well with the original function $f(t)$. This is the case where the data has been “ALIASED”.

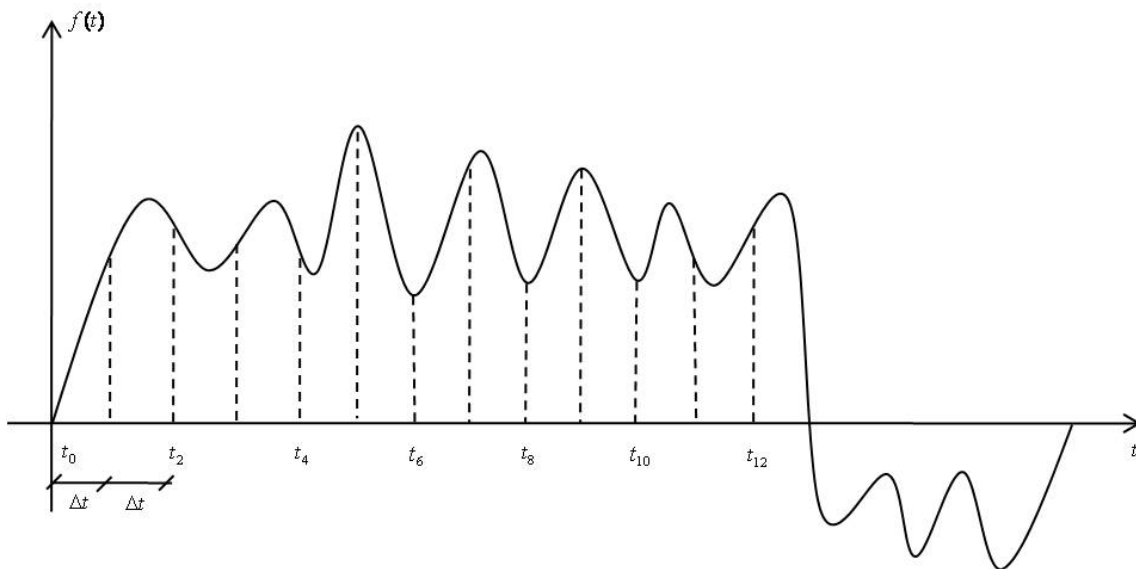


Figure 2 Function to be sampled and “Windowing” Sample Problem.

Another potential difficulty in sampling the function is called “windowing” problem. As indicated in Figure 2, while Δt is small enough so that a piecewise linear interpolation for connecting these discrete values will adequately resemble the original function $f(t)$, however, only a portion of the function $f(t)$ has been sampled (from t_1 through t_{12}) rather than the entire one. In other words, one has placed a “window” over the function.

To avoid aliased phenomenon, the sample space Δt should be small enough so that the discrete sequence will recover back the original function $f(t)$. The “sampling theorem” can be stated as:

“If the function $f(t)$ is band-limited with bandwidth $2w_{\max}$, $F(w) \equiv$ Fourier transform of $f(t) = 0$ for $|w| \geq w_{\max} > 0$ then $f(t)$ is uniquely determined by a knowledge of its values at uniformly spaced intervals Δt apart, with $\Delta t = \frac{1}{2w_{\max}}$.

The above “sampling theorem” can be loosely explained through the help of Figure 3.

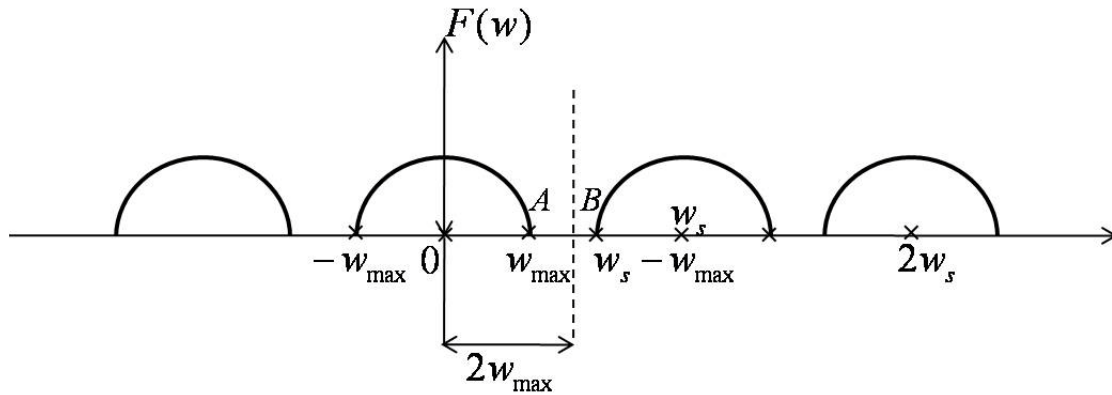


Figure 3 Frequency of sampling rate (w_s) versus maximum frequency content (w_{\max}).

To satisfy $F(w) = 0$, for $|w| \geq w_{\max}$, the frequency (w) should be between points A and B of Figure 3.

Hence

$$w_{\max} \leq w \leq w_s - w_{\max}$$

which implies

$$w_s \geq 2w_{\max}$$

Physically, the above equation states that one must have at least 2 samples per cycle of the highest frequency component present (Nyquist samples, Nyquist rate).

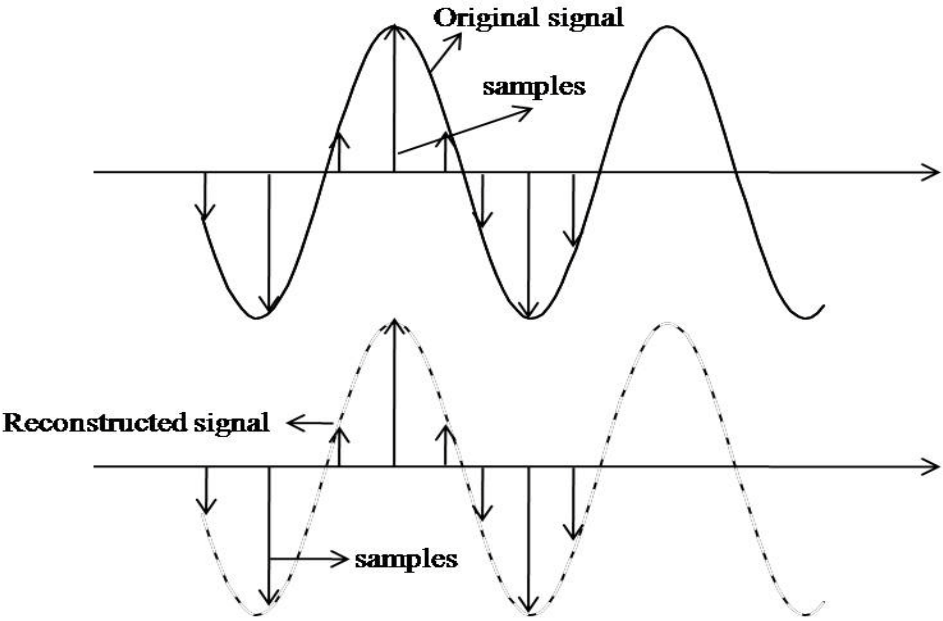


Figure 4 Correctly reconstructed signal.

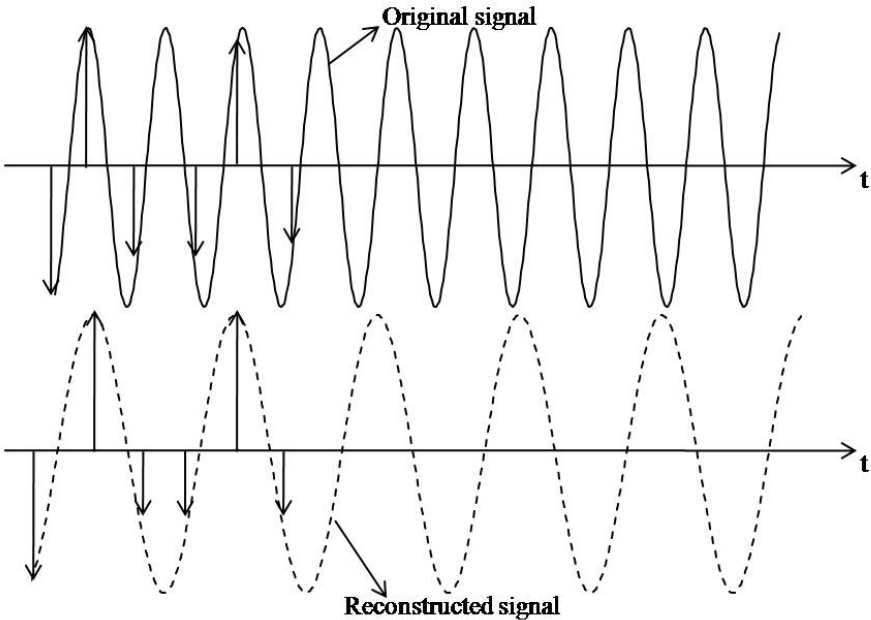


Figure 5 Wrongly reconstructed signal.

In Figure 4, a sinusoidal signal is sampled at the rate of 6 samples per 1 cycle (or $w_s = 6w_0$). Since this sampling rate does satisfy the sampling theorem requirement ($w_s \geq 2w_{\max}$), the reconstructed signal does correctly represent the original signal. However, as indicated in Figure 5 a sinusoidal signal is sampled at the rate of 6 samples per 4 cycles ($or w_s = \frac{6}{4}w_0$). Since this sampling rate does NOT satisfy the requirement ($w_s \geq 2w_{\max}$), the reconstructed signal would wrongly represent the original signal.

References

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FAST FOURIER TRANSFORM

Topic	Discrete Fourier Transform
Summary	Textbook notes on discrete Fourier transform
Major	General Engineering
Authors	Duc Nguyen
Date	July 25, 2010
Web Site	http://numericalmethods.eng.usf.edu
