Chapter 07.03

Simpson’s 1/3 Rule of Integration

After reading this chapter, you should be able to
1. derive the formula for Simpson’s 1/3 rule of integration,
2. use Simpson’s 1/3 rule it to solve integrals,
3. develop the formula for multiple-segment Simpson’s 1/3 rule of integration,
4. use multiple-segment Simpson’s 1/3 rule of integration to solve integrals, and
5. derive the true error formula for multiple-segment Simpson’s 1/3 rule.

What is integration?
Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson’s 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson’s 1/3 rule of approximating integrals of the form

\[ I = \int_{a}^{b} f(x) \, dx \]

where
\[ f(x) \] is called the integrand,
\[ a = \text{lower limit of integration} \]
\[ b = \text{upper limit of integration} \]

Simpson’s 1/3 Rule
The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson’s 1/3 rule is an
extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Figure 1 Integration of a function

Method 1:
Hence

\[ I = \int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} f_2(x) \, dx \]

where \( f_2(x) \) is a second order polynomial given by

\[ f_2(x) = a_0 + a_1x + a_2x^2. \]

Choose

\( (a, f(a)), \left( \frac{a+b}{2}, f\left( \frac{a+b}{2} \right) \right) \), and \( (b, f(b)) \)

as the three points of the function to evaluate \( a_0, a_1 \) and \( a_2 \).

\[
\begin{align*}
 f(a) &= f_2(a) = a_0 + a_1 a + a_2 a^2 \\
 f\left( \frac{a+b}{2} \right) &= f_2\left( \frac{a+b}{2} \right) = a_0 + a_1 \left( \frac{a+b}{2} \right) + a_2 \left( \frac{a+b}{2} \right)^2 \\
 f(b) &= f_2(b) = a_0 + a_1 b + a_2 b^2
\end{align*}
\]

Solving the above three equations for unknowns, \( a_0, a_1 \) and \( a_2 \) give

\[
\begin{align*}
a_0 &= \frac{a^2 f(b) + abf(b) - 4abf\left( \frac{a+b}{2} \right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2} \\
a_1 &= -\frac{af(a) - 4af\left( \frac{a+b}{2} \right) + 3af(b) + 3bf(a) - 4bf\left( \frac{a+b}{2} \right) + bf(b)}{a^2 - 2ab + b^2}
\end{align*}
\]
\[ a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2} \]

Then

\[ I \approx \int_{a}^{b} f_2(x)\,dx \]

\[ = \int_{a}^{b} \left(\frac{a_0x + a_1x + a_2x^2}{2} \right)\,dx \]

\[ = \left[ a_0x + a_1x^2 + a_2x^3 \right]_{a}^{b} \]

\[ = a_0(b-a) + a_1\frac{b^2 - a^2}{2} + a_2\frac{b^3-a^3}{3} \]

Substituting values of \(a_0\), \(a_1\), and \(a_2\) give

\[ \int_{a}^{b} f_2(x)\,dx = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \]

Since for Simpson 1/3 rule, the interval \([a, b]\) is broken into 2 segments, the segment width

\[ h = \frac{b-a}{2} \]

Hence the Simpson’s 1/3 rule is given by

\[ \int_{a}^{b} f(x)\,dx \approx \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \]

Since the above form has 1/3 in its formula, it is called Simpson’s 1/3 rule.

**Method 2:**
Simpson’s 1/3 rule can also be derived by approximating \(f(x)\) by a second order polynomial using Newton’s divided difference polynomial as

\[ f_2(x) = b_0 + b_1(x - a) + b_2(x - a)\left(x - \frac{a+b}{2}\right) \]

where

\[ b_0 = f(a) \]

\[ b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} \]

\[ b_2 = \frac{\frac{2}{3}b - \frac{1}{3}a}{4} + \frac{\frac{2}{3}a - \frac{1}{3}b}{4} \]

\[ b_3 = \frac{\frac{2}{3}a - \frac{1}{3}b}{4} - \frac{\frac{2}{3}b - \frac{1}{3}a}{4} \]

\[ \int_{a}^{b} f(x)\,dx \approx \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \]

Since the above form has 1/3 in its formula, it is called Simpson’s 1/3 rule.
Integrating Newton’s divided difference polynomial gives us

\[
\int_a^b f(x) \, dx \approx \int_a^b f_2(x) \, dx
\]

\[
= \int_a^b \left[ b_0 + b_1(x - a) + b_2(x - a) \left( x - \frac{a + b}{2} \right) \right] \, dx
\]

\[
= \left. \left[ b_0x + b_1 \left( \frac{x^2}{2} - ax \right) + b_2 \left( \frac{x^3}{3} - \frac{(3a + b)x^2}{4} + \frac{a(a + b)x}{2} \right) \right] \right|_a^b
\]

\[
= b_0(b - a) + b_1 \left( \frac{b^2 - a^2}{2} - a(b - a) \right)
\]

\[
+ b_2 \left( \frac{b^3 - a^3}{3} - \frac{(3a + b)(b^2 - a^2)}{4} + \frac{a(a + b)(b - a)}{2} \right)
\]

Substituting values of \( b_0, \ b_1, \) and \( b_2 \) into this equation yields the same result as before

\[
\int_a^b f(x) \, dx \approx \frac{b - a}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right]
\]

\[
= \frac{h}{3} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right]
\]

**Method 3:**
One could even use the Lagrange polynomial to derive Simpson’s formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

\[
f_2(x) = \frac{\left( x - \frac{a + b}{2} \right)(x - b)}{\left( a - \frac{a + b}{2} \right)(a - b)} \, f(a) + \frac{\left( x - a \right)(x - b)}{\left( \frac{a + b}{2} - a \right) \left( \frac{a + b}{2} - b \right)} \, f \left( \frac{a + b}{2} \right) + \frac{\left( x - a \right)}{\left( b - a \right) \left( b - \frac{a + b}{2} \right)} \, f(b)
\]

Integrating this function gets
Simpson’s 1/3 Rule of Integration

\[
\int_a^b f(x)dx = \left[ \frac{x^3 - (a + 3b)x^2 + b(a + b)x}{4} + \frac{b(a + b)}{2} \right] + f(a) \frac{a - a + b}{2} (a - b) + f(b) \frac{(b - a)(b - a + b)}{2} f(a)
\]

\[
\int_a^b f(x)dx = \left[ \frac{x^3 - (a + b)x^2 + abx}{3} + \frac{a + b}{2} \right] + f\left( \frac{a + b}{2} \right) f(b)
\]

Believe it or not, simplifying and factoring this large expression yields you the same result as before

\[
\int_a^b f(x)dx \approx \frac{b - a}{6} \left[ f(a) + 4f\left( \frac{a + b}{2} \right) + f(b) \right]
\]

Let the right-hand side be an exact expression for the integrals \( \int_a^b dx, \int_a^b xdx, \) and \( \int_a^b x^2dx \). This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

\[
\int_a^b 1dx = b - a = c_1 + c_2 + c_3
\]
\[
\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left( \frac{a+b}{2} \right)^2 + c_3 b^2
\]

Solving the above three equations for \(c_0\), \(c_1\) and \(c_2\) give
\[
\begin{align*}
  c_1 &= \frac{b-a}{6} \\
  c_2 &= \frac{2(b-a)}{3} \\
  c_3 &= \frac{b-a}{6}
\end{align*}
\]

This gives
\[
\int_a^b f(x) \, dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f \left( \frac{a+b}{2} \right) + \frac{b-a}{6} f(b)
\]
\[
= \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
\]
\[
= \frac{h}{3} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
\]

The integral from the first method
\[
\int_a^b f(x) \, dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2) \, dx
\]
can be viewed as the area under the second order polynomial, while the equation from Method 4
\[
\int_a^b f(x) \, dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f \left( \frac{a+b}{2} \right) + \frac{b-a}{6} f(b)
\]
can be viewed as the sum of the areas of three rectangles.

Example 1

The distance covered by a rocket in meters from \(t = 8\) s to \(t = 30\) s is given by
\[
x = \int_8^{30} \left( 2000 \ln \left( \frac{140000}{140000 - 2100 t} \right) - 9.8 t \right) dt
\]

a) Use Simpson’s 1/3 rule to find the approximate value of \(x\).
b) Find the true error, \(E_t\).
c) Find the absolute relative true error, \(|\epsilon|\).
Simpson’s 1/3 Rule of Integration

Solution

a) \[ x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \]
\[ a = 8 \]
\[ b = 30 \]
\[ \frac{a+b}{2} = 19 \]
\[ f(t) = 2000\ln\left(\frac{1400000}{140000 - 2100t}\right) - 9.8t \]
\[ f(8) = 2000\ln\left(\frac{1400000}{1400000 - 2100(8)}\right) - 9.8(8) = 177.27 \text{ m/s} \]
\[ f(30) = 2000\ln\left(\frac{1400000}{1400000 - 2100(30)}\right) - 9.8(30) = 901.67 \text{ m/s} \]
\[ f(19) = 2000\ln\left(\frac{1400000}{1400000 - 2100(19)}\right) - 9.8(19) = 484.75 \text{ m/s} \]
\[ x = \frac{b-a}{6} \left[ f(8) + 4f(19) + f(30) \right] \]
\[ = \frac{30-8}{6} \left[ 177.27 + 4 \times 484.75 + 901.67 \right] \]
\[ = 11065.72 \text{ m} \]

b) The exact value of the above integral is
\[ x = \int_{8}^{30} \left[ 2000\ln\left(\frac{1400000}{1400000 - 2100t}\right) - 9.8t \right] dt \]
\[ = 11061.34 \text{ m} \]

So the true error is
\[ E_t = \text{True Value} - \text{Approximate Value} \]
\[ = 11061.34 - 11065.72 \]
\[ = -4.38 \text{ m} \]

c) The absolute relative true error is
\[ \frac{\text{True Error}}{\text{True Value}} \times 100 \]
\[ = \frac{-4.38}{11061.34} \times 100 \]
Multiple-segment Simpson’s 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval \([a, b]\) into \(n\) segments and apply Simpson’s 1/3 rule repeatedly over every two segments. Note that \(n\) needs to be even. Divide interval \([a, b]\) into \(n\) equal segments, so that the segment width is given by
\[
h = \frac{b - a}{n}.
\]

Now
\[
\int_{a}^{b} f(x) \, dx = \int_{x_0}^{x_n} f(x) \, dx
\]
where
\[
x_0 = a,
\]
\[
x_n = b.
\]
\[
\int_{a}^{b} f(x) \, dx = \int_{x_0}^{x_2} f(x) \, dx + \int_{x_2}^{x_4} f(x) \, dx + \ldots + \int_{x_{n-2}}^{x_n} f(x) \, dx
\]
Apply Simpson’s 1/3rd Rule over each interval,
\[
\int_{a}^{b} f(x) \, dx \approx (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots
\]
\[
+ (x_{n-2} - x_{n-4}) \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]
\]
Since
\[
x_i - x_{i-2} = 2h,
\]
i = 2, 4, ..., \(n\)
then
\[
\int_{a}^{b} f(x) \, dx \approx 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots
\]
\[
+ 2h \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]
\]
\[
= \frac{h}{3} \left[ f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n) \right]
\]
Simpson’s 1/3 Rule of Integration

\[
\int_{a}^{b} f(x) \, dx \approx \frac{b-a}{3n} \left[ f(x_0) + 4 \sum_{i=1 \text{ odd}}^{n-1} f(x_i) + 2 \sum_{i=2 \text{ even}}^{n-2} f(x_i) + f(x_n) \right]
\]

Example 2

Use 4-segment Simpson’s 1/3 rule to approximate the distance covered by a rocket in meters from \( t = 8 \, \text{s} \) to \( t = 30 \, \text{s} \) as given by

\[
x = \int_{8}^{30} 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \, dt
\]

a) Use four segment Simpson’s 1/3rd Rule to estimate \( x \).
b) Find the true error, \( E_t \), for part (a).
c) Find the absolute relative true error, \( \varepsilon_t \), for part (a).

Solution:

a) Using \( n \) segment Simpson’s 1/3 rule,

\[
x \approx \frac{b-a}{3n} \left[ f(t_0) + 4 \sum_{i=1 \text{ odd}}^{n-1} f(t_i) + 2 \sum_{i=2 \text{ even}}^{n-2} f(t_i) + f(t_n) \right]
\]

\[
n = 4, \quad a = 8, \quad b = 30
\]

\[
h = \frac{b-a}{n} = \frac{30-8}{4} = 5.5
\]

\[
f(t) = 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t
\]

So

\[
f(t_0) = f(8)
\]

\[
f(8) = 2000 \ln \left( \frac{140000}{140000 - 2100(8)} \right) - 9.8(8) = 177.27 \, \text{m/s}
\]

\[
f(t_i) = f(8 + 5.5) = f(13.5)
\]
\begin{align*}
f(13.5) &= 2000 \ln \left( \frac{140000}{140000 - 2100(13.5)} \right) - 9.8(13.5) = 320.25 m/s \\
f(t_2) &= f(13.5 + 5.5) = f(19) \\
f(19) &= 2000 \ln \left( \frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 m/s \\
f(t_3) &= f(19 + 5.5) = f(24.5) \\
f(24.5) &= 2000 \ln \left( \frac{140000}{140000 - 2100(24.5)} \right) - 9.8(24.5) = 676.05 m/s \\
f(t_n) &= f(t_n) = f(30) \\
f(30) &= 2000 \ln \left( \frac{140000}{140000 - 2100(30)} \right) - 9.8(30) = 901.67 m/s \\
\end{align*}

\[
x = \frac{b - a}{3n} \left[ f(t_0) + 4 \sum_{i=1 \atop i=\text{odd}}^{n-1} f(t_i) + 2 \sum_{i=2 \atop i=\text{even}}^{n-2} f(t_i) + f(t_n) \right] \\
= \frac{30 - 8}{3(4)} \left[ f(8) + 4 f(t_1) + 4 f(t_3) + 2 f(t_2) + f(30) \right] \\
= \frac{22}{12} \left[ f(8) + 4 f(13.5) + 4 f(24.5) + 2 f(19) + f(30) \right] \\
= \frac{11}{6} \left[ 177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67 \right] \\
= 11061.64 \text{ m} \\
\]

b) The exact value of the above integral is

\[
x = \int_{8}^{30} \left( 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right) dt \\
= 11061.34 \text{ m} \\
\]

So the true error is

\[
E_t = \text{True Value} - \text{Approximate Value} \\
E_t = 11061.34 - 11061.64 \\
= -0.30 \text{ m} 
\]
c) The absolute relative true error is
\[ |\varepsilon_i| = \frac{|\text{True Error}|}{\text{True Value}} \times 100 \]
\[ = \frac{|-0.3|}{11061.34} \times 100 \\
= 0.0027\% \]

### Table 1  Values of Simpson’s 1/3 rule for Example 2 with multiple-segments

| n   | Approximate Value | \( E_i \) | \( |\varepsilon_i| \) |
|-----|------------------|----------|------------------|
| 2   | 11065.72         | -4.38    | 0.0396%          |
| 4   | 11061.64         | -0.30    | 0.0027%          |
| 6   | 11061.40         | -0.06    | 0.0005%          |
| 8   | 11061.35         | -0.02    | 0.0002%          |
| 10  | 11061.34         | -0.01    | 0.0001%          |

### Error in Multiple-segment Simpson’s 1/3 rule

The true error in a single application of Simpson’s 1/3rd Rule is given by

\[ E_i = \frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b \]

In multiple-segment Simpson’s 1/3 rule, the error is the sum of the errors in each application of Simpson’s 1/3 rule. The error in the \( n \) segments Simpson’s 1/3rd Rule is given by

\[
E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \\
= -\frac{h^5}{90} f^{(4)}(\zeta_1) \\
E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \\
= -\frac{h^5}{90} f^{(4)}(\zeta_2) \\
\vdots \\
E_i = -\frac{(x_{2i} - x_{2i-2})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2i-2} < \zeta_i < x_{2i} \\
= -\frac{h^5}{90} f^{(4)}(\zeta_i) \\
\vdots \\
E_{2n} = -\frac{(x_0 - x_{2n-1})^5}{2880} f^{(4)}(\zeta_{2n}), \quad x_{2n-1} < \zeta_{2n} < x_0 \\
= -\frac{h^5}{90} f^{(4)}(\zeta_{2n}) \\
\]

\[ ^1 \text{The } f^{(4)} \text{ in the true error expression stands for the fourth derivative of the function } f(x). \]
\[ E_{n-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)} \left( \zeta_{n-1} \right), \quad x_{n-4} < \zeta_{n-1} < x_{n-2} \]

\[ = -\frac{h^5}{90} f^{(4)} \left( \zeta_{n-1} \right) \]

\[ E_n = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)} \left( \zeta_n \right), \quad x_{n-2} < \zeta_n < x_n \]

Therefore, the total error in the multiple-segment Simpson’s 1/3 rule is

\[ E_i = \sum_{i=1}^{n} E_i \]

\[ = -\frac{h^5}{90} \sum_{i=1}^{n} f^{(4)} (\zeta_i) \]

\[ = -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{n} f^{(4)} (\zeta_i) \]

\[ = -\frac{(b-a)^5}{180n^4} \frac{n}{2}, \]

The term \( \frac{n}{2} \) is an approximate average value of \( f^{(4)}(x) \), \( a < x < b \). Hence

\[ E_i = -\frac{h^5}{90} \sum_{i=1}^{n} f^{(4)} (\zeta_i) \]

where

\[ \bar{f}^{(4)} = \frac{\sum_{i=1}^{n} f^{(4)} (\zeta_i)}{\frac{n}{2}} \]

**INTEGRATION**

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