Gauss Quadrature Rule of Integration

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Gauss Quadrature Rule of Integration

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What is Integration?

Integration

The process of measuring the area under a curve.

\[ I = \int_{a}^{b} f(x) \, dx \]

Where:
- \( f(x) \) is the integrand
- \( a \) = lower limit of integration
- \( b \) = upper limit of integration
Two-Point Gaussian Quadrature Rule
Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

\[
\int_{a}^{b} f(x) \, dx \approx c_1 f(a) + c_2 f(b)
\]

\[
= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)
\]
Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as $a$ and $b$, but as unknowns $x_1$ and $x_2$. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$ I = \int_{a}^{b} f(x) \, dx \approx c_1 f(x_1) + c_2 f(x_2) $$
Basis of the Gaussian Quadrature Rule

The four unknowns $x_1$, $x_2$, $c_1$ and $c_2$ are found by assuming that the formula gives exact results for integrating a general third order polynomial, 

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$ 

Hence

$$\int_a^b f(x) \, dx = \int_a^b \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 \right) \, dx$$

$$= \left[ a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b$$

$$= a_0 (b - a) + a_1 \left( \frac{b^2 - a^2}{2} \right) + a_2 \left( \frac{b^3 - a^3}{3} \right) + a_3 \left( \frac{b^4 - a^4}{4} \right)$$
Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1 \left(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3\right) + c_2 \left(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3\right)$$

Equating Equations the two previous two expressions yield

$$a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2}\right) + a_2 \left(\frac{b^3 - a^3}{3}\right) + a_3 \left(\frac{b^4 - a^4}{4}\right) = c_1 \left(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3\right) + c_2 \left(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3\right)$$

$$= a_0 (c_1 + c_2) + a_1 \left(c_1 x_1 + c_2 x_2\right) + a_2 \left(c_1 x_1^2 + c_2 x_2^2\right) + a_3 \left(c_1 x_1^3 + c_2 x_2^3\right)$$
Basis of the Gaussian Quadrature Rule

Since the constants $a_0$, $a_1$, $a_2$, $a_3$ are arbitrary

\[ b - a = c_1 + c_2 \]

\[ \frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2 \]

\[ \frac{b^3 - a^3}{3} = c_1 x_1^2 + c_2 x_2^2 \]

\[ \frac{b^4 - a^4}{4} = c_1 x_1^3 + c_2 x_2^3 \]
Basis of Gauss Quadrature

The previous four simultaneous nonlinear Equations have only one acceptable solution,

\[
x_1 = \left( \frac{b-a}{2} \right) \left( -\frac{1}{\sqrt{3}} \right) + \frac{b+a}{2}
\]

\[
x_2 = \left( \frac{b-a}{2} \right) \left( \frac{1}{\sqrt{3}} \right) + \frac{b+a}{2}
\]

\[
c_1 = \frac{b-a}{2}
\]

\[
c_2 = \frac{b-a}{2}
\]
Basis of Gauss Quadrature

Hence Two-Point Gaussian Quadrature Rule

\[ \int_{a}^{b} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) \]

\[
= \frac{b-a}{2} \left( \frac{b-a}{2} \left( -\frac{1}{\sqrt{3}} \right) + \frac{b+a}{2} \right) + \frac{b-a}{2} \left( \frac{b-a}{2} \left( \frac{1}{\sqrt{3}} \right) + \frac{b+a}{2} \right)
\]
Higher Point Gaussian Quadrature Formulas
Higher Point Gaussian Quadrature Formulas

\[ \int_{a}^{b} f(x) dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) \]

is called the three-point Gauss Quadrature Rule. The coefficients \( c_1, c_2, \) and \( c_3, \) and the functional arguments \( x_1, x_2, \) and \( x_3 \) are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

\[ \int_{a}^{b} \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \right) dx \]

General \( n \)-point rules would approximate the integral

\[ \int_{a}^{b} f(x) dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots + c_n f(x_n) \]
Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are as shown in Table 1.

\[ \int_{-1}^{1} g(x) dx \approx \sum_{i=1}^{n} c_i g(x_i) \]

as shown in Table 1.

<table>
<thead>
<tr>
<th>Points</th>
<th>Weighting Factors</th>
<th>Function Arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( c_1 = 1.000000000 ) ( c_2 = 1.000000000 )</td>
<td>( x_1 = -0.577350269 ) ( x_2 = 0.577350269 )</td>
</tr>
<tr>
<td>3</td>
<td>( c_1 = 0.555555556 ) ( c_2 = 0.888888889 ) ( c_3 = 0.555555556 )</td>
<td>( x_1 = -0.774596669 ) ( x_2 = 0.000000000 ) ( x_3 = 0.774596669 )</td>
</tr>
<tr>
<td>4</td>
<td>( c_1 = 0.347854845 ) ( c_2 = 0.652145155 ) ( c_3 = 0.652145155 ) ( c_4 = 0.347854845 )</td>
<td>( x_1 = -0.861136312 ) ( x_2 = -0.339981044 ) ( x_3 = 0.339981044 ) ( x_4 = 0.861136312 )</td>
</tr>
</tbody>
</table>
Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Table 1 (cont.) : Weighting factors $c$ and function arguments $x$ used in Gauss Quadrature Formulas.

<table>
<thead>
<tr>
<th>Points</th>
<th>Weighting Factors</th>
<th>Function Arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$c_1 = 0.236926885$</td>
<td>$x_1 = -0.906179846$</td>
</tr>
<tr>
<td></td>
<td>$c_2 = 0.478628670$</td>
<td>$x_2 = -0.538469310$</td>
</tr>
<tr>
<td></td>
<td>$c_3 = 0.568888889$</td>
<td>$x_3 = 0.000000000$</td>
</tr>
<tr>
<td></td>
<td>$c_4 = 0.478628670$</td>
<td>$x_4 = 0.538469310$</td>
</tr>
<tr>
<td></td>
<td>$c_5 = 0.236926885$</td>
<td>$x_5 = 0.906179846$</td>
</tr>
<tr>
<td>6</td>
<td>$c_1 = 0.171324492$</td>
<td>$x_1 = -0.932469514$</td>
</tr>
<tr>
<td></td>
<td>$c_2 = 0.360761573$</td>
<td>$x_2 = -0.661209386$</td>
</tr>
<tr>
<td></td>
<td>$c_3 = 0.467913935$</td>
<td>$x_3 = -0.2386191860$</td>
</tr>
<tr>
<td></td>
<td>$c_4 = 0.467913935$</td>
<td>$x_4 = 0.2386191860$</td>
</tr>
<tr>
<td></td>
<td>$c_5 = 0.360761573$</td>
<td>$x_5 = 0.661209386$</td>
</tr>
<tr>
<td></td>
<td>$c_6 = 0.171324492$</td>
<td>$x_6 = 0.932469514$</td>
</tr>
</tbody>
</table>
Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

So if the table is given for \( \int_{-1}^{1} g(x) \, dx \) integrals, how does one solve \( \int_{a}^{b} f(x) \, dx \)? The answer lies in that any integral with limits of \([a, b]\) can be converted into an integral with limits \([-1, 1]\). Let

\[
x = mt + c
\]

If \( x = a \), then \( t = -1 \)

If \( x = b \), then \( t = 1 \)

Such that:

\[
m = \frac{b - a}{2}
\]
Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Then \[ c = \frac{b + a}{2} \]

Hence

\[ x = \frac{b - a}{2} t + \frac{b + a}{2} \]
\[ dx = \frac{b - a}{2} dt \]

Substituting our values of \( x \), and \( dx \) into the integral gives us

\[ \int_{a}^{b} f(x) dx = \int_{-1}^{1} f\left( \frac{b - a}{2} t + \frac{b + a}{2} \right) \frac{b - a}{2} dt \]
Example 1

For an integral \( \int_{a}^{b} f(x) \, dx \), derive the one-point Gaussian Quadrature Rule.

Solution

The one-point Gaussian Quadrature Rule is

\[
\int_{a}^{b} f(x) \, dx \approx c_1 f(x_1)
\]
Solution

The two unknowns $x_{1}$, and $c_{1}$ are found by assuming that the formula gives exact results for integrating a general first order polynomial,

$$f(x) = a_{0} + a_{1}x.$$  

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} (a_{0} + a_{1}x)dx$$  

$$= \left[ a_{0}x + a_{1} \frac{x^{2}}{2} \right]_{a}^{b}$$  

$$= a_{0}(b - a) + a_{1}\left(\frac{b^{2} - a^{2}}{2}\right)$$
Solution

It follows that

\[ \int_{a}^{b} f(x)\,dx = c_1(a_0 + a_1x_1) \]

Equating Equations, the two previous two expressions yield

\[ a_0(b - a) + a_1\left(\frac{b^2 - a^2}{2}\right) = c_1(a_0 + a_1x_1) = a_0(c_1) + a_1(c_1x_1) \]
Basis of the Gaussian Quadrature Rule

Since the constants $a_0$, and $a_1$ are arbitrary

$$b - a = c_1$$

$$\frac{b^2 - a^2}{2} = c_1 x_1$$

giving

$$c_1 = b - a$$

$$x_1 = \frac{b + a}{2}$$
Solution

Hence One-Point Gaussian Quadrature Rule

\[
\int_a^b f(x)dx \approx c_1 f(x_1) = (b - a) f\left(\frac{b + a}{2}\right)
\]
Example 2

a) Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from $t=8$ to $t=30$ as given by

$$x = \int_{8}^{30} \left( 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right) dt$$

b) Find the true error, $E_t$, for part (a).

c) Also, find the absolute relative true error, $|e_a|$, for part (a).
Solution

First, change the limits of integration from \([8,30]\) to \([-1,1]\) by previous relations as follows

\[
\int_{8}^{30} f(t) dt = \frac{30 - 8}{2} \int_{-1}^{1} f\left(\frac{30 - 8}{2} x + \frac{30 + 8}{2}\right) dx
\]

\[
= 11 \int_{-1}^{1} f(11x + 19) dx
\]
Next, get weighting factors and function argument values from Table 1 for the two point rule,

\[ c_1 = 1.000000000 \]
\[ x_1 = -0.577350269 \]
\[ c_2 = 1.000000000 \]
\[ x_2 = 0.577350269 \]
Solution (cont.)

Now we can use the Gauss Quadrature formula

\[ 11 \int_{-1}^{1} f(11x + 19) \, dx \approx 11c_1 f(11x_1 + 19) + 11c_2 f(11x_2 + 19) \]

\[ = 11f(11(-0.5773503) + 19) + 11f(11(0.5773503) + 19) \]

\[ = 11f(12.64915) + 11f(25.35085) \]

\[ = 11(296.8317) + 11(708.4811) \]

\[ = 11058.44 \, m \]
Solution (cont)

since

\[ f(12.64915) = 2000 \ln \left[ \frac{140000}{140000 - 2100(12.64915)} \right] - 9.8(12.64915) \]

\[ = 296.8317 \]

\[ f(25.35085) = 2000 \ln \left[ \frac{140000}{140000 - 2100(25.35085)} \right] - 9.8(25.35085) \]

\[ = 708.4811 \]
Solution (cont)

b) The true error, $E_t$, is

$$E_t = True \ Value - Approximate \ Value$$

$$= 11061.34 - 11058.44$$

$$= 2.9000 \ m$$

c) The absolute relative true error, $|\varepsilon_t|$, is (Exact value = 11061.34m)

$$|\varepsilon_t| = \left| \frac{11061.34 - 11058.44}{11061.34} \right| \times 100\%$$

$$= 0.0262\%$$
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/gauss_quadrature.html
THE END

http://numericalmethods.eng.usf.edu