Chapter 04.01
Introduction to Matrix Algebra

After reading this chapter, you should be able to

1. *know what a matrix is,*
2. *identify special types of matrices,*
3. *know when two matrices are equal,*
4. *add, subtract and multiply matrices,*
5. *rules of binary matrix operations,*
6. *find transpose of a matrix,*
7. *find inverse of a matrix and its application to solving simultaneous linear equations.*

What is a matrix?

Matrices are everywhere. If you have used a spreadsheet such as Excel or written a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program. Look at the matrix below about the sale of tires in a Blowoutr'us store – given by quarter and make of tires.

$$\begin{array}{l} \\ \textbf{Tirestone} \\ \textbf{Michigan} \\ \textbf{Copper} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Quarter 1} & \text{Quarter 2} & \text{Quarter 3} & \text{Quarter 4} \end{array} \\ \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \end{array}$$

If one wants to know how many *Copper* tires were sold in *Quarter 4*, we go along the row *Copper* and column *Quarter 4* and find that it is 27.

So what is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix [A] is denoted by

$$[A] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}$$

Row i of [A] has n elements and is $\begin{bmatrix} a_{i1} & a_{i2} .... a_{in} \end{bmatrix}$ and

Column j of [A] has m elements and is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Each matrix has rows and columns and this defines the size of the matrix. If a matrix [A] has m rows and n columns, the size of the matrix is denoted by $m \times n$. The matrix [A] may also be denoted by $[A]_{m\times n}$ to show that $[A]$ is a matrix with m rows and n columns.

Each entry in the matrix is called the entry or element of the matrix and is denoted by a_{ij} where i is the row number and j is the column number of the element.
The matrix for the tire sales example could be denoted by the matrix [A] as

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

There are 3 rows and 4 columns, so the size of the matrix is 3×4. In the above $[A]$ matrix, $a_{34} = 27$.

What are the special types of matrices?

Vector: A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.
Row vector: If a matrix has one row, it is called a row vector

$$[B] = [b_1 \ b_2 \ \ldots\ldots b_m]$$

and m is the dimension of the row vector.

Example 1

Give an example of a row vector.
Solution

[B] = [25 20 3 2 0] is an example of a row vector of dimension 5.

Column vector: If a matrix has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

and n is the dimension of the vector.

Example 2

Give an example of a column vector.

Solution

$$[C]=\begin{bmatrix}25\\5\\6\end{bmatrix}$$

is an example of a column vector of dimension 3.

Square matrix: If the number of rows (m) of a matrix is equal to the number of columns (n) of the matrix, ($m = n$), it is called a square matrix. The entries $a_{11}, a_{22}, ... a_{nn}$ are called the diagonal elements of a square matrix. Sometimes the diagonal of the matrix is also called the principal or main of the matrix.

Example 3

Give an example of a square matrix.

Solution

$$[A]=\begin{bmatrix}25 & 20 & 3\\5 & 10 & 15\\6 & 15 & 7\end{bmatrix}$$

is a square matrix as it has same number of rows and columns, that is, three.
The diagonal elements of [A] are $a_{11} = 25, a_{22} = 10, a_{33} = 7$.

Upper triangular matrix: A $m \times n$ matrix for which $a_{ij} = 0, i > j$ is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

Example 4

Give an example of an upper triangular matrix.

Solution

$$[A]=\begin{bmatrix}10 & -7 & 0\\0 & -0.001 & 6\\0 & 0 & 15005\end{bmatrix}$$

is an upper triangular matrix.

Lower triangular matrix: A $m \times n$ matrix for which $a_{ij} = 0, j > i$ is called a lower triangular matrix. That is, all the elements above the diagonal entries are zero.

Example 5

Give an example of a lower triangular matrix.

Solution

$$[A]=\begin{bmatrix}1 & 0 & 0\\ 0.3 & 1 & 0\\ 0.6 & 2.5 & 1\end{bmatrix}$$

is a lower triangular matrix.

Diagonal matrix: A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, ($a_{ij}=0,\ i\neq j$).

Example 6

Give examples of a diagonal matrix.
Solution

$$[A]=\begin{bmatrix}3 & 0 & 0\\ 0 & 2.1 & 0\\ 0 & 0 & 5\end{bmatrix}$$

is a diagonal matrix.
Any or all the diagonal entries of a diagonal matrix can be zero.
For example

$$[A]=\begin{bmatrix}3 & 0 & 0\\ 0 & 2.1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

is also a diagonal matrix.

Identity matrix: A diagonal matrix with all diagonal elements equal to one is called an identity matrix, ($a_{ij}=0,\ i\neq j$; and $a_{ii}=1$ for all i).

Example 7

Give an example of an identity matrix.
Solution

$$[A]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

is an identity matrix.

Zero matrix: A matrix whose all entries are zero is called a zero matrix, ($a_{ij}=0$ for all i and j).

Example 8

Give examples of a zero matrix.
Solution

$$[A]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$[B]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$[C]=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$
$$[D]=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

are all examples of a zero matrix.

Tridiagonal matrices: A tridiagonal matrix is a square matrix in which all elements not on the major diagonal, the diagonal above the major diagonal and the diagonal below the major diagonal are zero.

Example 9

Give an example of a tridiagonal matrix.
Solution

$$[A]=\begin{bmatrix}2 & 4 & 0 & 0\\ 2 & 3 & 9 & 0\\ 0 & 0 & 5 & 2\\ 0 & 0 & 3 & 6\end{bmatrix}$$

is a tridiagonal matrix.

When are two matrices considered to be equal?

Two matrices [A] and [B] are equal if the size of [A] and [B] is the same (number of rows and columns are same for [A] and [B]) and $a_{ij}=b_{ij}$ for all i and j.

Example 10

What would make

$$[A]=\begin{bmatrix}2 & 3\\ 6 & 7\end{bmatrix}$$ to be equal to

$$[B]=\begin{bmatrix}b_{11} & 3\\ 6 & b_{22}\end{bmatrix},$$

Solution

The two matrices $[A]$ and $[B]$ would be equal if
$b_{11} = 2, b_{22} = 7$.

How do you add two matrices?

Two matrices [A] and [B] can be added only if they are the same size, then the addition is shown as

$$[C] = [A] + [B]$$

where

$$c_{ij} = a_{ij} + b_{ij}$$

Example 11

Add two matrices

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Solution

$$[C] = [A] + [B]$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6 & 2+7 & 3-2 \\ 1+3 & 2+5 & 7+19 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 1 \\ 4 & 7 & 26 \end{bmatrix}$$

Example 12

Blowout r’us store has two locations A and B, and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number - 1, 2, 3, 4. What are the total sales of the two locations by make and quarter?

Solution

$$[C]=[A]+[B]$$
$$=\begin{bmatrix}25 & 20 & 3 & 2\\ 5 & 10 & 15 & 25\\ 6 & 16 & 7 & 27\end{bmatrix}+\begin{bmatrix}20 & 5 & 4 & 0\\ 3 & 6 & 15 & 21\\ 4 & 1 & 7 & 20\end{bmatrix}$$
$$=\begin{bmatrix}(25+20) & (20+5) & (3+4) & (2+0)\\ (5+3) & (10+6) & (15+15) & (25+21)\\ (6+4) & (16+1) & (7+7) & (27+20)\end{bmatrix}$$
$$=\begin{bmatrix}45 & 25 & 7 & 2\\ 8 & 16 & 30 & 46\\ 10 & 17 & 14 & 47\end{bmatrix}$$

So if one wants to know the total number of Copper tires sold in quarter 4 in the two locations, we would look at Row 3 – Column 4 to give

$$c_{34}=47.$$

How do you subtract two matrices?

Two matrices [A] and [B] can be subtracted only if they are the same size and the subtraction is given by

$$[D] = [A] - [B]$$

where

$$d_{ij}=a_{ij}-b_{ij}$$

Example 13

Subtract matrix [B] from matrix [A].

$$[A]=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix}$$
$$[B]=\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}$$

Solution

$$[C]=[A]-[B]$$
$$=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix}-\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}$$
$$=\begin{bmatrix}5-6 & 2-7 & 3-(-2)\\ 1-3 & 2-5 & 7-19\end{bmatrix}$$
$$=\begin{bmatrix}-1 & -5 & 5\\ -2 & -3 & -12\end{bmatrix}$$

Example 14

Blowout r'us store has two locations A and B and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[\mathrm{A}] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[\mathrm{B}] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number- 1, 2, 3, 4. How many more tires did store A sell than store B of each brand in each quarter?

Solution

$$[\mathrm{D}] = [\mathrm{A}] - [\mathrm{B}]$$

$$= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} - \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 25-20 & 20-5 & 3-4 & 2-0 \\ 5-3 & 10-6 & 15-15 & 25-21 \\ 6-4 & 16-1 & 7-7 & 27-20 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 15 & -1 & 2 \\ 2 & 4 & 0 & 4 \\ 2 & 15 & 0 & 7 \end{bmatrix}$$

So if you want to know how many more Copper Tires were sold in quarter 4 in Store A than Store B, $d_{34} = 7$. Note that $d_{13} = -1$ implying that store A sold 1 less Michigan tire than Store B in quarter 3.

How do I multiply two matrices?

Two matrices [A] and [B] can be multiplied only if the number of columns of [A] is equal to the number of rows of [B] to give

$$[\mathrm{C}]_{m\times n} = [\mathrm{A}]_{m\times p}[\mathrm{B}]_{p\times n}$$

If [A] is a $m \times p$ matrix and [B] is a $p \times n$ matrix, the resulting matrix [C] is a $m \times n$ matrix.

So how does one calculate the elements of [C] matrix?

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots\ldots + a_{ip} b_{pj}$$

for each $i = 1,\ 2, \ldots\ldots, m$, and $j = 1,\ 2, \ldots\ldots, n$.

To put it in simpler terms, the i_{th} row and j_{th} column of the [C] matrix in [C] = [A][B] is calculated by multiplying the i_{th} row of [A] by the j_{th} column of [B], that is,

$$c_{ij} = \left[a_{i1}\ a_{i2} \ldots\ldots a_{ip}\right] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{bmatrix}$$

$$= a_{i1}\, b_{1j} \ + \ a_{i2}\, b_{2j} + \ \ldots\ldots + \ a_{ip}\, b_{pj}.$$

$$= \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 15

Given

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix}$$

find

$$[C] = [A][B]$$

Solution

c_{12} can be found by multiplying the first row of [A] by the second column of [B],

$$c_{12} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$

$$= -56$$

Similarly, one can find the other elements of [C] to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

Example 16

Blowout r’us store location A and the sales of tires are given by make (in rows) and quarters (in columns) as shown below

$$[A]=\begin{bmatrix}25 & 20 & 3 & 2\\ 5 & 10 & 15 & 25\\ 6 & 16 & 7 & 27\end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number - 1, 2, 3, 4. Find the per quarter sales of store A if following are the prices of each tire.

Tirestone = $33.25

Michigan = $40.19

Copper = $25.03

Solution

The answer is given by multiplying the price matrix by the quantity sales of store A. The price matrix is $[33.25 \quad 40.19 \quad 25.03]$, then the per quarter sales of store A would be given by

$$[C]=[33.25 \quad 40.19 \quad 25.03]\begin{bmatrix}25 & 20 & 3 & 2\\ 5 & 10 & 15 & 25\\ 6 & 16 & 7 & 27\end{bmatrix}$$

$$c_{ij}=\sum_{k=1}^{3} a_{ik}b_{kj}$$

$$\begin{aligned}c_{11}&=\sum_{k=1}^{3} a_{1k}b_{k1}\\ &=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}\\ &=(33.25)(25)+(40.19)(5)+(25.03)(6)\\ &=\$1182.38\end{aligned}$$

Similarly

$$c_{12}=\$1467.38,$$

$$c_{13}=\$877.81,$$

$$c_{14}=\$1747.06.$$

So each quarter sales of store A in dollars are given by the four columns of the row vector

$$[C]=[1182.38 \quad 1467.38 \quad 877.81 \quad 1747.06]$$

Remember since we are multiplying a 1×3 matrix by a 3×4 matrix, the resulting matrix is a 1×4 matrix.

What is a scalar product of a constant and a matrix?

If $[A]$ is a $n\times n$ matrix and k is a real number, then the scalar product of k and [A] is another matrix [B], where $b_{ij}=k\,a_{ij}$.

Example 17

Let $[A]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}$. Find 2 [A]

Solution

$$[A]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}$$

Then

$$2[A]=2\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}$$
$$=\begin{bmatrix}(2)(2.1) & (2)(3) & (2)(2)\\ (2)(5) & (2)(1) & (2)(6)\end{bmatrix}$$
$$=\begin{bmatrix}4.2 & 6 & 4\\ 10 & 2 & 12\end{bmatrix}$$

What is a linear combination of matrices?

If $[A_1], [A_2], ..., [A_p]$ are matrices of the same size and $k_1, k_2, ..., k_p$ are scalars, then

$$k_1[A_1]+k_2[A_2]+.......+k_p[A_p]$$

is called a linear combination of $[A_1], [A_2], \cdots\cdots, [A_p]$.

Example 18

If

$$[A_1]=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}, [A_2]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}, [A_3]=\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix}$$

then find

$$[A_1]+2[A_2]-0.5[A_3]$$

Solution

$$=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+2\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}-0.5\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+\begin{bmatrix}4.2 & 6 & 4\\ 10 & 2 & 12\end{bmatrix}-\begin{bmatrix}0 & 1.1 & 1\\ 1.5 & 1.75 & 3\end{bmatrix}$$
$$=\begin{bmatrix}9.2 & 10.9 & 5\\ 11.5 & 2.25 & 10\end{bmatrix}$$

What are some of the rules of binary matrix operations?

Commutative law of addition

If [A] and [B] are mxn matrices, then

$$[A]+[B]=[B]+[A]$$

Associate law of addition

If [A], [B] and [C] all are mxn matrices, then

$$[A]+([B]+[C])=([A]+[B])+[C]$$

Associate law of multiplication

If [A], [B] and [C] are mxn, nxp and pxr size matrices, respectively, then

$$[A]([B][C])=([A][B])[C]$$

and the resulting matrix size on both sides is mxr.

Distributive law

If [A] and [B] are mxn size matrices, and [C] and [D] are nxp size matrices

$$[A]([C]+[D])=[A][C]+[A][D]$$

$$([A]+[B])[C]=[A][C]+[B][C]$$

and the resulting matrix size on both sides is mxp.

Example 19

Illustrate the associative law of multiplication of matrices using

$$[A]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix},\quad [B]=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix},\quad [C]=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

Solution

$$[B][C]=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}=\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}$$

$$[A][B][C]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

$$[A][B]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}$$

$$[A][B][C]=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

The above illustrates the associate law of multiplication of matrices.

Is [A][B]=[B][A]?

First both operations $[A][B]$ and $[B][A]$ are only possible if $[A]$ and $[B]$ are square matrices of same size. Why? If $[A][B]$ exists, number of columns of $[A]$ has to be same as the number of rows of $[B]$ and if $[B][A]$ exists, number of columns of $[B]$ has to be same as the number of rows of $[A]$.

Even then in general $[A][B] \neq [B][A]$.

Example 20

Illustrate if $[A][B]=[B][A]$ for the following matrices

$$[A] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}, \quad [B] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

Solution

$$[A][B] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix}$$

$$[B][A] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}$$

$$[A][B] \neq [B][A]$$

Transpose of a matrix: Let [A] be a $m \times n$ matrix. Then [B] is the transpose of the [A] if $b_{ji} = a_{ij}$ for all i and j. That is, the i^{th} row and the j^{th} column element of [A] is the j^{th} row and i^{th} column element of [B]. Note, [B] would be a $n \times m$ matrix. The transpose of [A] is denoted by $[A]^t$.

Example 21

Find the transpose of

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

Solution

The transpose of [A] is

$$[\mathrm{A}]^T = \begin{bmatrix} 25 & 5 & 6 \\ 20 & 10 & 16 \\ 3 & 15 & 7 \\ 2 & 25 & 27 \end{bmatrix}$$

Note, the transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

Also, note that the transpose of a transpose of a matrix is the matrix itself, that is,

$$\left([\mathrm{A}]^T\right)^T = [\mathrm{A}]. \quad \text{Also, } ([\mathrm{A}]+[\mathrm{B}])^T = [\mathrm{A}]^T + [\mathrm{B}]^T; (c[\mathrm{A}])^T = c[\mathrm{A}]^T \quad .$$

Symmetric matrix: A square matrix [A] with real elements where $a_{ij} = a_{ji}$ for $i = 1, \ldots\ldots, n$ and $j = 1, \ldots\ldots, n$ is called a symmetric matrix. This is same as, if $[\mathrm{A}] = [\mathrm{A}]^T$, then $[\mathrm{A}]$ is a symmetric matrix.

Example 22

Give an example of a symmetric matrix.

Solution

$$[\mathrm{A}] = \begin{bmatrix} 21.2 & 3.2 & 6 \\ 3.2 & 21.5 & 8 \\ 6 & 8 & 9.3 \end{bmatrix}$$

is a symmetric matrix as $a_{12} = a_{21} = 3.2$; $\mathrm{a}_{13} = a_{31} = 6$ and $a_{23} = a_{32} = 8$.

Matrix algebra is used for solving system of equations. Can you illustrate this concept?

Matrix algebra is used to solve a system of simultaneous linear equations. In fact, for many mathematical procedures such as solution of set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations. Let us illustrate with an example for interpolation.

Example 23

The upward velocity of a rocket is given at three different times on the following table

Time, t	**Velocity,** v
s	m/s
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = at^2 + bt + c, \quad 5 \le t \le 12.$$

Set up the equations in matrix form to find the coefficients a, b, c of the velocity profile.

Solution

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2)$, and (t_3, v_3) where from the above table

$$t_1 = 5, v_1 = 106.8$$
$$t_2 = 8, v_2 = 177.2$$
$$t_3 = 12, v_3 = 279.2$$

Requiring that $v(t) = at^2 + bt + c$ passes through the three data points gives

$$v(t_1) = v_1 = at_1^2 + bt_1 + c$$
$$v(t_2) = v_2 = at_2^2 + bt_2 + c$$
$$v(t_3) = v_3 = at_3^2 + bt_3 + c$$

Substituting the data $(t_1, v_1), (t_2, v_2), (t_3, v_3)$ gives

$$a(5^2) + b(5) + c = 106.8$$
$$a(8^2) + b(8) + c = 177.2$$
$$a(12^2) + b(12) + c = 279.2$$

or

$$25a + 5b + c = 106.8$$
$$64a + 8b + c = 177.2$$
$$144a + 12b + c = 279.2$$

This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25a + & 5b + & c \\ 64a + & 8b + & c \\ 144a + & 12b + & c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above equation can be written as a linear combination as follows

$$a\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + b\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and further using matrix multiplications gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above is an illustration of why matrix algebra is needed. The complete solution to the set of equations is given later in this chapter.

For a general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{22}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_m \end{bmatrix}$$

Denoting the matrices by $[A]$, $[X]$, and $[C]$, the system of equation is $[A]\ [X]=[C]$, where $[A]$ is called the coefficient matrix, $[C]$ is called the right hand side vector and $[X]$ is called the solution vector.

Sometimes $[A]\ [X]=[C]$ systems of equations is written in the augmented form. That is

$$[A \vdots C] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots\, c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots\, c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots\, c_n \end{bmatrix}$$

Can you divide two matrices?

If [A] [B] = [C] is defined, it might seem intuitive that $[A]=\frac{[C]}{[B]}$, but matrix division is not defined. However an inverse of a matrix can be defined for certain types of square matrices. The inverse of a square matrix [A], if existing, is denoted by $[A]^{-1}$ such that $[A][A]^{-1} = [I] = [A]^{-1}[A]$.

In other words, let [A] be a square matrix. If [B] is another square matrix of same size such that [B][A] = [I], then [B] is the inverse of [A]. [A] is then called to be invertible or nonsingular. If $[A]^{-1}$ does not exist, [A] is called to be noninvertible or singular.

If [A] and [B] are two *nxn* matrices such that [B] [A] = [I], then these statements are also true

a) [B] is the inverse of [A]
b) [A] is the inverse of [B]
c) [A] and [B] are both invertible
d) [A] [B]=[I].
e) [A] and [B] are both nonsingular
f) all columns of [A] or [B]are linearly independent
g) all rows of [A] or [B] are linearly independent.

Example 24

Show if

$$[B]=\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix} \text{ is the inverse of } [A]=\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$$

Solution

$$[B][A]=\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$
$$=[I]$$

Since [B] [A] = [I], [B] is the inverse of [A] and [A] is the inverse of [B]. But we can also show that

$$[A][B]=\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=[I]$$

to show that [A] is the inverse of [B].

Can I use the concept of the inverse of a matrix to find the solution of a set of equations [A] [X] = [C]?

Yes, if the number of equations is same as the number of unknowns, the coefficient matrix [A] is a square matrix.

Given

$$[A][X]=[C]$$

Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$.

$$[A]^{-1}[A][X]=[A]^{-1}[C]$$
$$[I][X]=[A]^{-1}[C]$$
$$[X]=[A]^{-1}[C]$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of [A][X] = [C] is simply a multiplication of $[A]^{-1}$ and the right hand side vector, [C].

How do I find the inverse of a matrix?

If [A] is a $n\times n$ matrix, then $[A]^{-1}$ is a $n\times n$ matrix and according to the definition of inverse of a matrix

$[A][A]^{-1}=[I]$.

Denoting

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & a'_{nn} \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & & & & 0 \\ 0 & & \cdot & & & \cdot \\ \cdot & & & 1 & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Using the definition of matrix multiplication, the first column of the $[A]^{-1}$ matrix can then be found by solving

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ \cdot \\ \cdot \\ a'_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Similarly, one can find the other columns of the $[A]^{-1}$ matrix by changing the right hand side accordingly.

Example 25

The upward velocity of the rocket is given by

Time, t	**Velocity,** v
s	m/s
5	106.8
8	177.2
12	279.2

In an earlier example, we wanted to approximate the velocity profile by

$$v(t) = at^2 + bt + c, \quad 5 \le 8 \le 12$$

We found that the coefficients a, b, c are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

First find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and then use the definition of inverse to find the coefficients a, b, c.

Solution

If $[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$

is the inverse of [A],

Then

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives three sets of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the above three sets of equations separately gives

$$\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Now

$$[A][X] = [C]$$

where

$$[X] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the definition of $[A]^{-1}$,

$$[A]^{-1}[A][X] = [A]^{-1}[C]$$

$$[X] = [A]^{-1}[C]$$

$$= \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix} \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

So

$$v(t) = at^2 + bt + c, 5 \le \text{t} \le 12$$

$$= 0.2900t^2 + 19.70t + 1.050, 5 \le \text{t} \le 12$$

If the inverse of a square matrix [A] exists, is it unique?

Yes, the inverse of a square matrix is unique, if it exists. The proof is as follows. Assume that the inverse of [A] is [B] and if this inverse is not unique, then let another inverse of [A] exist called [C].

[B] is inverse of [A], then

[B][A] = [I]

Multiply both sides by [C],

[B][A][C] = [I][C]
[B][A][C] = [C]

Since [C] is inverse of [A], [A][C] = [I]

[B][I] = [C]
[B] = [C]

This shows that [B] and [C] are the same. So inverse of [A] is unique.

INTRODUCTION TO MATRIX ALGEBRA	
Topic	Introduction to Matrix Algebra
Summary	Know what a matrix is; Identify special types of matrices; When two matrices are equal; Add, subtract and multiply matrices; Learn rules of binary operations on matrices; Know what unary operations mean; Find the transpose of a square matrix and it relationship to symmetric matrices; Setup simultaneous linear equations in matrix form and vice-versa; Understand the concept of inverse of a matrix.
Major	General Engineering
Authors	Autar Kaw
Date	March 23, 2010
Web Site	http://numericalmethods.eng.usf.edu