Chapter 04.03
Binary Matrix Operations

After reading this chapter, you should be able to

1. *add, subtract, and multiply matrices, and*
2. *apply rules of binary operations on matrices.*

How do you add two matrices?

Two matrices $[A]$ and $[B]$ can be added only if they are the same size. The addition is then shown as

$$[C]=[A]+[B]$$

where

$$c_{ij}=a_{ij}+b_{ij}$$

Example 1

Add the following two matrices.

$$[A]=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix} \qquad [B]=\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}$$

Solution

$$[C]=[A]+[B]$$

$$=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix}+\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}$$

$$=\begin{bmatrix}5+6 & 2+7 & 3-2\\ 1+3 & 2+5 & 7+19\end{bmatrix}$$

$$=\begin{bmatrix}11 & 9 & 1\\ 4 & 7 & 26\end{bmatrix}$$

Example 2

Blowout r'us store has two store locations A and B, and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?

Solution

$$\begin{aligned}[C] &= [A] + [B] \\ &= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} + \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix} \\ &= \begin{bmatrix} (25+20) & (20+5) & (3+4) & (2+0) \\ (5+3) & (10+6) & (15+15) & (25+21) \\ (6+4) & (16+1) & (7+7) & (27+20) \end{bmatrix} \\ &= \begin{bmatrix} 45 & 25 & 7 & 2 \\ 8 & 16 & 30 & 46 \\ 10 & 17 & 14 & 47 \end{bmatrix}\end{aligned}$$

So if one wants to know the total number of Copper tires sold in quarter 4 at the two locations, we would look at Row 3 – Column 4 to give $c_{34} = 47$.

How do you subtract two matrices?

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size. The subtraction is then given by

$$[D] = [A] - [B]$$

Where

$$d_{ij} = a_{ij} - b_{ij}$$

Example 3

Subtract matrix $[B]$ from matrix $[A]$.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Solution

$$[D]=[A]-[B]$$

$$=\begin{bmatrix}5 & 2 & 3\\1 & 2 & 7\end{bmatrix}-\begin{bmatrix}6 & 7 & -2\\3 & 5 & 19\end{bmatrix}$$

$$=\begin{bmatrix}(5-6) & (2-7) & (3-(-2))\\(1-3) & (2-5) & (7-19)\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -5 & 5\\-2 & -3 & -12\end{bmatrix}$$

Example 4

Blowout r'us has two store locations A and B and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A]=\begin{bmatrix}25 & 20 & 3 & 2\\5 & 10 & 15 & 25\\6 & 16 & 7 & 27\end{bmatrix}$$

$$[B]=\begin{bmatrix}20 & 5 & 4 & 0\\3 & 6 & 15 & 21\\4 & 1 & 7 & 20\end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. How many more tires did store A sell than store B of each brand in each quarter?

Solution

$$[D]=[A]-[B]$$

$$=\begin{bmatrix}25 & 20 & 3 & 2\\5 & 10 & 15 & 25\\6 & 16 & 7 & 27\end{bmatrix}-\begin{bmatrix}20 & 5 & 4 & 0\\3 & 6 & 15 & 21\\4 & 1 & 7 & 20\end{bmatrix}$$

$$=\begin{bmatrix}25-20 & 20-5 & 3-4 & 2-0\\5-3 & 10-6 & 15-15 & 25-21\\6-4 & 16-1 & 7-7 & 27-20\end{bmatrix}$$

$$=\begin{bmatrix}5 & 15 & -1 & 2\\2 & 4 & 0 & 4\\2 & 15 & 0 & 7\end{bmatrix}$$

So if you want to know how many more Copper tires were sold in quarter 4 in store A than store B, $d_{34}=7$. Note that $d_{13}=-1$ implies that store A sold 1 less Michigan tire than store B in quarter 3.

How do I multiply two matrices?

Two matrices $[A]$ and $[B]$ can be multiplied only if the number of columns of $[A]$ is equal to the number of rows of $[B]$ to give

$$[C]_{m\times n} = [A]_{m\times p}[B]_{p\times n}$$

If $[A]$ is a $m\times p$ matrix and $[B]$ is a $p\times n$ matrix, the resulting matrix $[C]$ is a $m\times n$ matrix.

So how does one calculate the elements of $[C]$ matrix?

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$
$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{ip}b_{pj}$$

for each $i = 1,\ 2,\ldots\ldots, m$ and $j = 1,\ 2,\ldots\ldots, n$.

To put it in simpler terms, the i^{th} row and j^{th} column of the $[C]$ matrix in $[C]=[A][B]$ is calculated by multiplying the i^{th} row of $[A]$ by the j^{th} column of $[B]$, that is,

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \ldots\ldots & a_{ip} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{bmatrix}$$
$$= a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ldots. + a_{ip}\, b_{pj}.$$
$$= \sum_{k=1}^{p} a_{ik}b_{kj}$$

Example 5

Given

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix}$$

Find

$$[C] = [A][B]$$

Solution

c_{12} can be found by multiplying the first row of $[A]$ by the second column of $[B]$,

$$c_{12} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$
$$= -56$$

Similarly, one can find the other elements of $[C]$ to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

Example 6

Blowout r'us store location A and the sales of tires are given by make (in rows) and quarters (in columns) as shown below

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. Find the per quarter sales of store A if the following are the prices of each tire.

Tirestone = \$33.25
Michigan = \$40.19
Copper = \$25.03

Solution

The answer is given by multiplying the price matrix by the quantity of sales of store A. The price matrix is $[33.25 \quad 40.19 \quad 25.03]$, so the per quarter sales of store A would be given by

$$[C] = [33.25 \quad 40.19 \quad 25.03] \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$$

$$c_{11} = \sum_{k=1}^{3} a_{1k} b_{k1}$$
$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$
$$= (33.25)(25) + (40.19)(5) + (25.03)(6)$$
$$= \$1182.38$$

Similarly

$$c_{12} = \$1467.38$$
$$c_{13} = \$877.81$$
$$c_{14} = \$1747.06$$

Therefore, each quarter sales of store A in dollars is given by the four columns of the row vector

$$[C] = [1182.38 \quad 1467.38 \quad 877.81 \quad 1747.06]$$

Remember since we are multiplying a 1×3 matrix by a 3×4 matrix, the resulting matrix is a 1×4 matrix.

What is the scalar multiplication of a matrix?

If $[A]$ is a $m \times n$ matrix and k is a real number, then the multiplication $[A]$ by a scalar k is another $m \times n$ matrix $[B]$, where

$$b_{ij} = k\,a_{ij} \text{ for all } i, j.$$

Example 7

Let

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

Find $2[A]$

Solution

$$2[A] = 2\begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 2\times 2.1 & 2\times 3 & 2\times 2 \\ 2\times 5 & 2\times 1 & 2\times 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix}$$

What is a linear combination of matrices?

If $[A_1],[A_2],.....,[A_p]$ are matrices of the same size and $k_1, k_2,.....,k_p$ are scalars, then

$$k_1[A_1] + k_2[A_2] + ... + k_p[A_p]$$

is called a linear combination of $[A_1],[A_2],...,[A_p]$.

Example 8

If $[A_1] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix}, [A_2] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}, [A_3] = \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}$

find

$$[A_1] + 2[A_2] - 0.5[A_3]$$

Solution

$$[A_1] + 2[A_2] - 0.5[A_3]$$
$$= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} - 0.5\begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 1.1 & 1 \\ 1.5 & 1.75 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 9.2 & 10.9 & 5 \\ 11.5 & 2.25 & 10 \end{bmatrix}$$

What are some of the rules of binary matrix operations?

Commutative law of addition

If $[A]$ and $[B]$ are $m\times n$ matrices, then

$$[A]+[B]=[B]+[A]$$

Associative law of addition

If [A], [B] and [C] are all $m\times n$ matrices, then

$$[A]+([B]+[C])=([A]+[B])+[C]$$

Associative law of multiplication

If $[A]$, $[B]$ and $[C]$ are $m\times n, n\times p$ and $p\times r$ size matrices, respectively, then

$$[A]([B][C])=([A][B])[C]$$

and the resulting matrix size on both sides of the equation is $m\times r$.

Distributive law

If $[A]$ and $[B]$ are $m\times n$ size matrices, and $[C]$ and $[D]$ are $n\times p$ size matrices

$$[A]([C]+[D])=[A][C]+[A][D]$$

$$([A]+[B])[C]=[A][C]+[B][C]$$

and the resulting matrix size on both sides of the equation is $m\times p$.

Example 9

Illustrate the associative law of multiplication of matrices using

$$[A]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix},\quad [B]=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix},\quad [C]=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

Solution

$$[B][C]=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

$$=\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}$$

$$[A]([B][C])=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}$$

$$=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

$$[A][B] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix}$$

$$([A][B])[C] = \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}$$

The above illustrates the associative law of multiplication of matrices.

Is [A][B] = [B][A]?

If $[A]\,[B]$ exists, number of columns of $[A]$ has to be same as the number of rows of $[B]$ and if $[B][A]$ exists, number of columns of $[B]$ has to be same as the number of rows of $[A]$. Now for $[A][B]=[B][A]$, the resulting matrix from $[A][B]$ and $[B][A]$ has to be of the same size. This is only possible if $[A]$ and $[B]$ are square and are of the same size. Even then in general $[A][B] \neq [B][A]$

Example 10

Determine if

$$[A][B] = [B][A]$$

for the following matrices

$$[A] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}, \quad [B] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

Solution

$$[A][B] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix}$$

$$[B][A] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}$$

$$[A][B] \neq [B][A]$$

Key Terms:

Addition of matrices
Subtraction of matrices
Multiplication of matrices
Scalar Product of matrices
Linear Combination of Matrices
Rules of Binary Matrix Operation