Chapter 03.07
Newton-Raphson Method of Solving Simultaneous Nonlinear Equations

After reading this chapter, you should be able to:

1. *derive the Newton-Raphson method formula for simultaneous nonlinear equations,*
2. *develop the algorithm of the Newton-Raphson method for solving simultaneous nonlinear equations,*
3. *use the Newton-Raphson method to solve a set of simultaneous nonlinear equations,*
4. *model a real-life problem that results in a set of simultaneous nonlinear equations.*

Introduction

Several physical systems result in a mathematical model in terms of simultaneous nonlinear equations. A set of such equations can be written as

$$\begin{aligned} &f_1(x_1, x_2, ..., x_n) = 0 \\ &... \\ &... \\ &f_n(x_1, x_2, ..., x_n) = 0 \end{aligned} \tag{1}$$

The solution to these simultaneous nonlinear equations are values of $x_1, x_2, ..., x_n$ which satisfy all the above n equations. The number of set of solutions to these equations could be none, unique, more than one but finite, or infinite. In this chapter, we use the Newton-Raphson method to solve these equations.

The Newton-Raphson method of solving a single nonlinear equation, $f(x) = 0$, can be derived using first-order Taylor series (first two terms of Taylor series) and are given by

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) \tag{2}$$

where

$x_i =$ previous estimate of root

$x_{i+1} =$ present estimate of root

Since we are looking for x_{i+1} where $f(x_{i+1})$ becomes zero, Equation (2) can be re-written as

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

and then as a recursive formula as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{3}$$

Derivation

Now how do we extend the same to simultaneous nonlinear equations? For sake of simplicity, let us limit the number of nonlinear equations to two as

$$u(x, y) = 0 \tag{4a}$$

$$v(x, y) = 0 \tag{4b}$$

The first order Taylor-series for nonlinear equation is (4a) & (4b) are

$$u(x_{i+1}, y_{i+1}) = u(x_i, y_i) + \frac{\partial u}{\partial x}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{\partial u}{\partial y}\Big|_{x_i, y_i}(y_{i+1} - y_i) \tag{5a}$$

$$v(x_{i+1}, y_{i+1}) = v(x_i, y_i) + \frac{\partial v}{\partial x}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{\partial v}{\partial y}\Big|_{x_i, y_i}(y_{i+1} - y_i) \tag{5b}$$

We are looking for (x_{i+1}, y_{i+1}) where $u(x_{i+1}, y_{i+1})$ and $v(x_{i+1}, y_{i+1})$ are zero. Hence

$$0 = u(x_i, y_i) + \frac{\partial u}{\partial x}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{\partial u}{\partial y}\Big|_{x_i, y_i}(y_{i+1} - y_i) \tag{6a}$$

$$0 = v(x_i, y_i) + \frac{\partial v}{\partial x}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{\partial v}{\partial y}\Big|_{x_i, y_i}(y_{i+1} - y_i) \tag{6b}$$

Writing

$$x_{i+1} - x_i = \Delta x \tag{7a}$$

$$y_{i+1} - y_i = \Delta y \tag{7b}$$

we get

$$0 = u(x_i, y_i) + \frac{\partial u}{\partial x}\Big|_{x_i, y_i}\Delta x + \frac{\partial u}{\partial y}\Big|_{x_i, y_i}\Delta y \tag{8a}$$

$$0 = v(x_i, y_i) + \frac{\partial v}{\partial x}\Big|_{x_i, y_i}\Delta x + \frac{\partial v}{\partial y}\Big|_{x_i, y_i}\Delta y \tag{8b}$$

Rewriting Equations (8a) and (8b)

$$\frac{\partial u}{\partial x}\Big|_{x_i, y_i}\Delta x + \frac{\partial u}{\partial y}\Big|_{x_i, y_i}\Delta y = -u(x_i, y_i) \tag{9a}$$

$$\frac{\partial v}{\partial x}\Big|_{x_i, y_i}\Delta x + \frac{\partial v}{\partial y}\Big|_{x_i, y_i}\Delta y = -v(x_i, y_i) \tag{9b}$$

and then in the matrix form

$$\begin{bmatrix} \frac{\partial u}{\partial x}\Big|_{x_i, y_i} & \frac{\partial u}{\partial y}\Big|_{x_i, y_i} \\ \frac{\partial v}{\partial x}\Big|_{x_i, y_i} & \frac{\partial v}{\partial y}\Big|_{x_i, y_i} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -u(x_i, y_i) \\ -v(x_i, y_i) \end{bmatrix} \tag{10}$$

Solving Equation (10) would give us Δx and Δy. Since the previous estimate of the root is (x_i, y_i), one can find from Equation (7)

$$x_{i+1} = x_i + \Delta x \tag{11a}$$

$$y_{i+1} = y_i + \Delta y \tag{11b}$$

This process is repeated till one obtains the root of the equation within a prespecified tolerance, ε_s such that

$$\left|\varepsilon_a\right|_x = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100 < \varepsilon_s$$

$$\left|\varepsilon_a\right|_y = \left|\frac{y_{i+1} - y_i}{y_{i+1}}\right| \times 100 < \varepsilon_s$$

Example 1

Find the roots of the simultaneous nonlinear equations

$$x^2 + y^2 = 50$$

$$x - y = 10$$

Use an initial guess of $(x, y) = (2, -4)$. Conduct two iterations.

Solution

First put the equations in the form

$$u(x, y) = 0$$

$$v(x, y) = 0$$

to give

$$u(x, y) = x^2 + y^2 - 50 = 0$$

$$v(x, y) = x - y - 10 = 0$$

Now

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial x} = 1$$

$$\frac{\partial v}{\partial y} = -1$$

Hence from Equation (10)

$$\begin{bmatrix} \left.\dfrac{\partial u}{\partial x}\right|_{x_i, y_i} & \left.\dfrac{\partial u}{\partial y}\right|_{x_i, y_i} \\ \left.\dfrac{\partial v}{\partial x}\right|_{x_i, y_i} & \left.\dfrac{\partial v}{\partial y}\right|_{x_i, y_i} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -u(x_i, y_i) \\ -v(x_i, y_i) \end{bmatrix}$$

$$\begin{bmatrix} 2x_i & 2y_i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -x_i^2 - y_i^2 + 50 \\ -x_i + y_i + 10 \end{bmatrix}$$

<u>Iteration 1</u>
The initial guess

$$(x_i, y_i) = (2, -4)$$

Hence

$$\begin{bmatrix} 2(2) & 2(-4) \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -(2)^2 - (-4)^2 + 50 \\ -(2) + (-4) + 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -8 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 30 \\ 4 \end{bmatrix}$$

Solving the equations by any method of your choice, we get

$$\Delta x = 0.5000$$

$$\Delta y = -3.500$$

Since

$$\Delta x = x_2 - x_1 = 0.5000$$

$$\Delta y = y_2 - y_1 = -3.500$$

we get

$$\begin{aligned} x_2 &= x_1 + 0.5000 \\ &= 2 + 0.5000 \\ &= 2.5000 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + (-3.500) \\ &= -4 + (-3.500) \\ &= -7.500 \end{aligned}$$

The absolute relative approximate errors at the end of the first iteration are

$$\begin{aligned} \left|\epsilon_a\right|_x &= \left|\frac{x_2 - x_1}{x_2}\right| \times 100 \\ &= \left|\frac{2.500 - 2.000}{2.500}\right| \times 100 \\ &= 20.00\% \end{aligned}$$

$$|\varepsilon_a|_y = \left|\frac{y_2 - y_1}{y_2}\right| \times 100$$
$$= \left|\frac{-7.500 - (-4.000)}{-7.500}\right| \times 100$$
$$= 46.67\%$$

Iteration 2

The estimate of the root at the end of iteration#1 is

$$(x_2, y_2) = (2.500 - 7.500)$$

Hence

$$\begin{bmatrix} 2(2.500) & 2(-7.500) \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -(2.500)^2 - (-7.500)^2 + 50 \\ -(2.500) + (-7.500) + 10 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -15 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -12.50 \\ 0 \end{bmatrix}$$

Solving the above equations by any method of your choice gives

$$\Delta x = 1.250$$
$$\Delta y = 1.250$$

Since

$$\Delta x = x_3 - x_2 = 1.250$$
$$\Delta y = y_3 - y_2 = 1.250$$

giving

$$x_3 = x_2 + 1.250$$
$$= 2.500 + 1.250$$
$$= 3.750$$
$$y_3 = y_2 + 1.250$$
$$= -7.500 + 1.250$$
$$= -6.250$$

The absolute relative approximate error at the end of the second iteration is

$$|\in_a|_x = \left|\frac{x_3 - x_2}{x_3}\right| \times 100$$
$$= \left|\frac{3.750 - 2.5}{3.750}\right| \times 100$$
$$= 33.33\%$$

$$
\begin{aligned}
\left|\in_a\right|_y &= \left|\frac{y_3 - y_2}{y_3}\right| \times 100 \\
&= \left|\frac{-6.250-(-7.500)}{-6.250}\right| \times 100 \\
&= 20.00\%
\end{aligned}
$$

Although not asked in the example problem statement, the estimated values of the root and the absolute relative approximate errors are given below in Table 1.

Table 1: Estimate of the root and absolute relative approximate error.

Iteration number, i	x_i	y_i	$\left\|\in_a\right\|_x \%$	$\left\|\in_a\right\|_y \%$
1	2.500	-7.500	20.00	46.67
2	3.750	-6.250	33.33	20.00
3	4.375	-5.625	14.29	11.11
4	4.688	-5.312	6.667	5.882
5	4.844	-5.156	3.226	3.030
6	4.922	-5.078	1.587	1.538

The exact solution to which the above scheme is converging to is $(x, y) = (5, -5)$

Example 2

A 5m long gutter is made from a flat sheet of aluminum which is 5m$\times$0.21m. The shape of the gutter cross-section is shown in Figure 1, and is made by bending the sheet at two locations at an angle θ (Figure 2). What are the values of s and θ, that will maximize the volume capacity of the gutter so that it drains water quickly during a heavy rainfall?

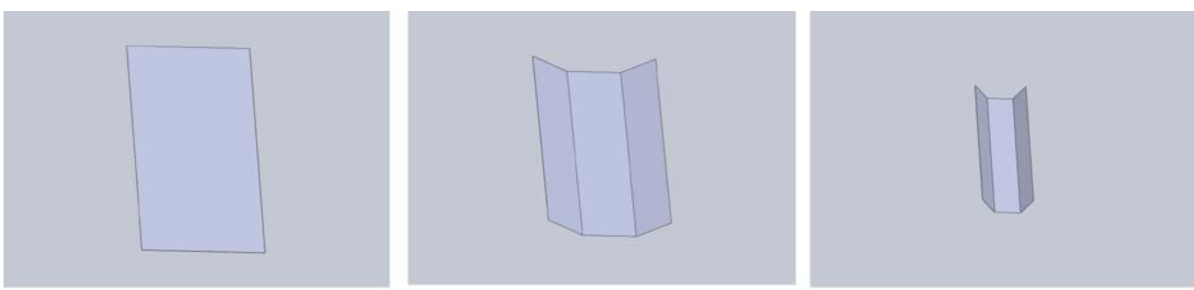

Figure 1 Sheet metal bent to form a gutter

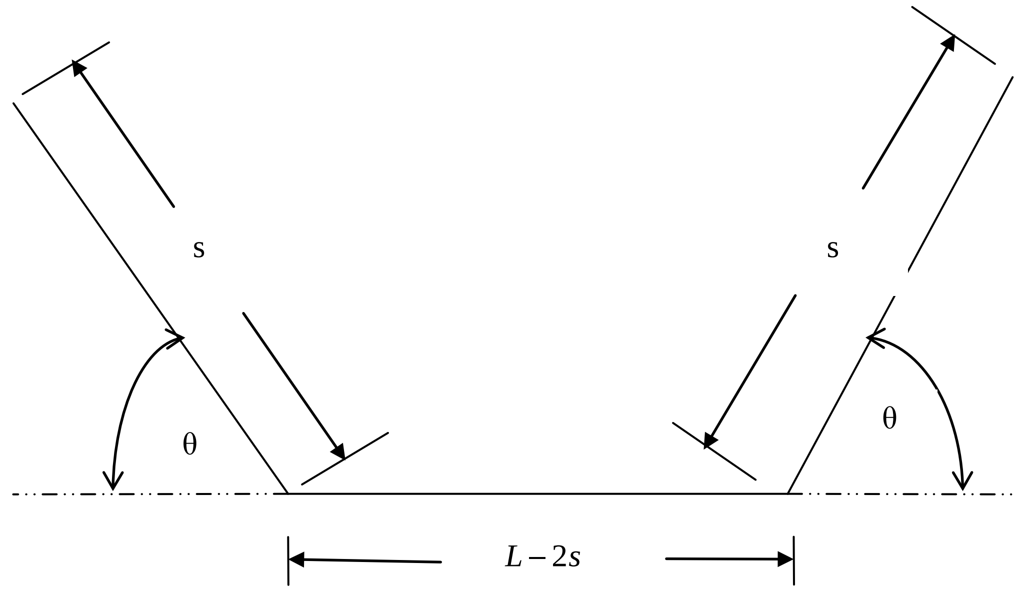

Figure 2: Parameters of the gutter

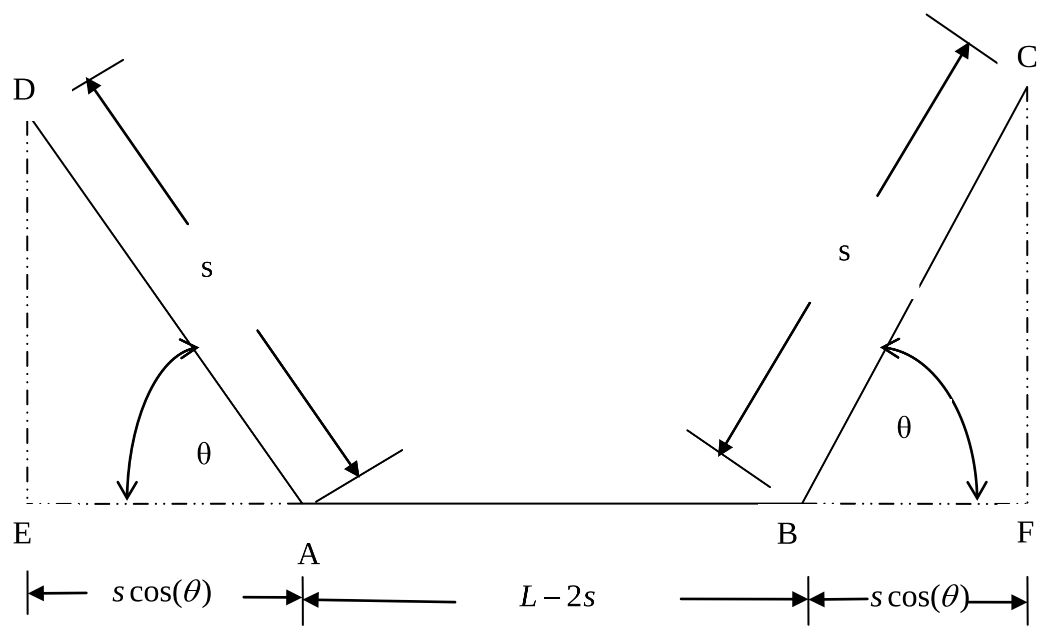

Figure 3: Labeling the parameters and points of the gutter

Solution

One needs to maximize the cross-sectional area of the gutter. The cross-sectional area G of the gutter is given by

$$G = \text{Area of trapezoid } ABCD$$
$$= \frac{1}{2}(AB + CD)(FC) \qquad \text{(E2.1)}$$

where

$$AB = L - 2\,s \qquad \text{(E2.2)}$$
$$CD = AB + EA + BF$$
$$= (L-2s) + s\cos(\theta) + s\cos(\theta)$$
$$= (L-2s) + 2s\cos(\theta) \qquad \text{(E2.3)}$$
$$FC = BC\ \sin(\theta)$$
$$= s\sin(\theta) \qquad \text{(E2.4)}$$

hence

$$G = \frac{1}{2}(AB + CD)\ (DE)$$
$$G(s,\theta) = \frac{1}{2}\left[(L-2s) + (L-2s) + 2s\cos(\theta)\right] s\ \sin(\theta)$$
$$= \left[L - 2s + s\cos(\theta)\right] s\sin(\theta) \qquad \text{(E2.5)}$$

For example, for $\theta = 0°$, the cross-sectional area of the gutter becomes

$$G = 0$$

as it represents a flat sheet, for $\theta = 180°$, the cross-sectional area of the gutter becomes

$$G = 0$$

as it represents an overlapped sheet, and for $\theta = 90°$, it represents a rectangular cross-sectional area with

$$G = (L-2s)\ s\ .$$

For the given L =0.21m,

$$G(s,\theta) = (0.21 - 2s + s\cos\theta)s\sin\theta$$

Then

$$\frac{\partial G}{\partial s} = -s^2\sin^2\theta + (0.21 - 2s + s\cos\theta)s\cos\theta$$
$$\frac{\partial G}{\partial \theta} = (-2 + \cos\theta)s\sin\theta + (0.21 + 2s + s\cos\theta)\sin\theta$$

By solving the equations

$$f_1(s,\theta) = -s\sin^2\theta + (0.21 - 2s + s\cos\theta)s\cos\theta = 0$$
$$f_2(s,\theta) = (-2 + \cos\theta)s\sin\theta + (0.21 - 2s + s\cos\theta)\sin\theta = 0$$

we can find the local minimas and maximas of $G(s,\theta)$. One of the local maximas may also be the absolute maximum. The values of s and θ that correspond to the maximum of $G(s,\theta)$ are what we are looking for. Can you solve the equations to find the corresponding values of s and θ for maximum value of G? Intuitively, what do you think would be these values of s and θ?

Appendix A: General matrix form of solving simultaneous nonlinear equations.

The general system of equations is given by

$$
\begin{aligned}
&f_1(x_1, x_2, ..., x_n) = 0\\
&f_2(x_1, x_2, ..., x_n) = 0\\
&...\\
&...\\
&...\\
&f_n(x_1, x_2, ..., x_n) = 0
\end{aligned}
\qquad (A.1)
$$

We can rewrite in a form as

$$F(\hat{x}) = \hat{0} \qquad (A.2)$$

where

$$F(\hat{x}) = \begin{bmatrix} f_1(\hat{x}) \\ f_2(\hat{x}) \\ . \\ . \\ f_n(\hat{x}) \end{bmatrix} \qquad (A.3)$$

$$\hat{0} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \qquad (A.4)$$

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}^T \qquad (A.5)$$

The Jacobian of the system of equations by using Newton-Raphson method then is

$$J(x_1, x_2, ..., x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{A.6}$$

Using the Jacobian for the Newton-Raphson method guess

$$[J]\,\Delta\hat{x} = -F(\hat{x}) \tag{A.7}$$

where

$$\Delta\hat{x} = \hat{x}_{new} - \hat{x}_{old} \tag{A.8}$$

Hence

$$\Delta\hat{x} = -[J]^{-1} F(\hat{x})$$

$$\hat{x}_{new} - \hat{x}_{old} = -[J]^{-1} F(\hat{x})$$

$$\hat{x}_{new} = \hat{x}_{old} - [J]^{-1} F(\hat{x}) \tag{A.10}$$

Evaluating the inverse of $[J]$ in Equation (A.10) is computationally more intensive than solving Equation (A.7) for $\Delta\hat{x}$ and calculating $\hat{x}_{new}$ as

$$\hat{x}_{new} = \hat{x}_{old} + \Delta\hat{x} \tag{A.11}$$