Chapter 04.00E

Physical Problem for Electrical Engineering Simultaneous Linear Equations

Problem Statement

Three-phase AC systems are the norm for most industrial applications. AC power in the form of voltage and current it delivered from the power company using three-phase distribution systems and many larger loads are three-phase loads in the form of motors, compressors, or similar. Sources and loads can be configured in either wye (where sources or loads are connected from line to neutral/ground) or delta (where sources or loads are connected from line to line) configurations and mixing between the types is common. Figure 1 shows the general wiring of a wye-wye three-phase system modeling all of the impedances typically found in such a system.

During the typical analysis undertaken in most circuits textbooks, it is assumed that the system is entirely balanced. This means that all the source, line, and load impedances are equivalent, that is,

$$Z_a = Z_b = Z_c$$

$$Z_{aA} = Z_{bB} = Z_{cC}$$

$$Z_{AN} = Z_{bN} = Z_{CN}$$

Under this assumption, the circuit is then typically reduced to a single-phase equivalent circuit model and the resultant circuit is solved with a single loop equation. What happens, however, when the system is unbalanced? Typically because the three load impedances Z_{AN}, Z_{BN} and Z_{CN} are not equal, which results in different currents through each load, is often measured in terms of the percentage difference between the load currents.

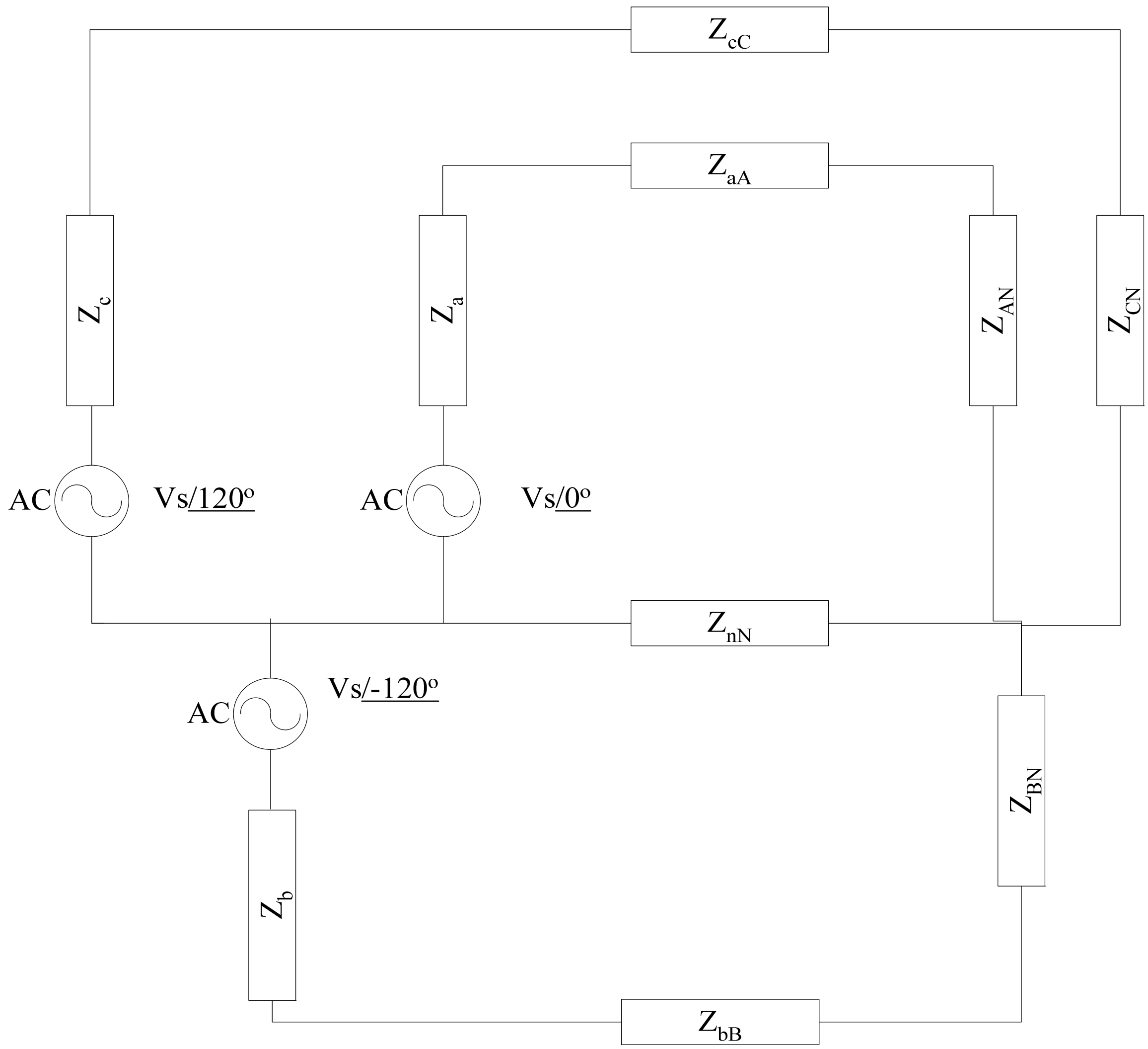

Figure 1 A Three-Phase Wye-Wye System with Positive Phase Sequence

Creating an imbalance in a three-phase system is not all that difficult. Consider a small business operating in an isolated leg of the power grid so that localized aspects of a load are not "balanced" by other neighboring loads. Let's assume that the primary load for this system is a 45 kVA set of three-phase motors at 0.8 power factor lagging and, further, that the electrician that did the wiring for the lighting mistakenly connected two banks of lights to the A phase, one to the B phase and none to the C phase creating an imbalance in the system. Each of these lighting loads is 1500 W. The load for this system is shown in Figure 2.

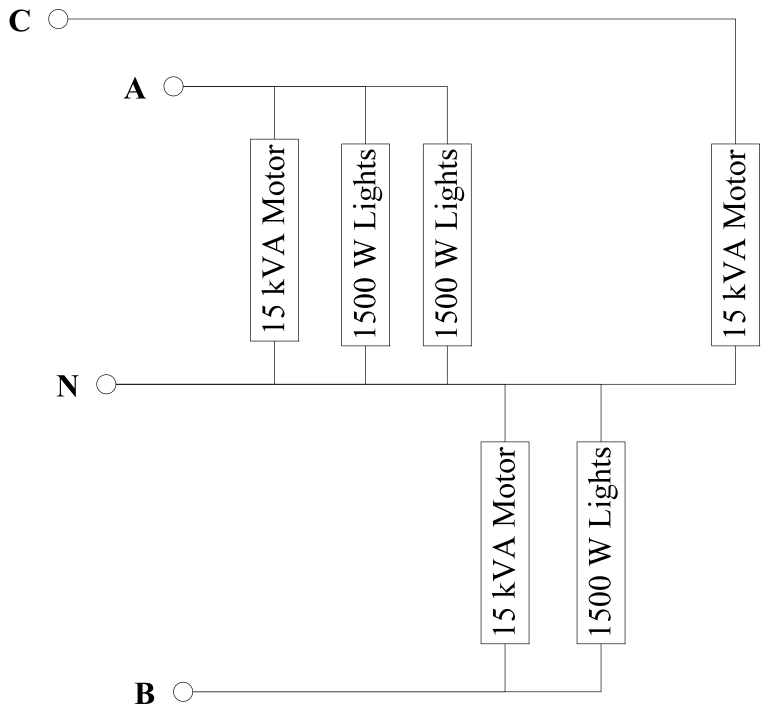

Figure 2 Model of the System Load

The impedance of each of the loads can be determined by examining the power consumed in each phase of the system

$$A: 3000 + 15000\,/\underline{-36.87^o} = 3000 + 12000 - j9000 = 15000 - j9000 = 17.49\,/\underline{-30.96^o}kVA$$

$$B: 1500 + 15000\,/\underline{-36.87^o} = 1500 + 12000 - j9000 = 13500 - j9000 = 16.22\,/\underline{-33.69^o}kVA$$

$$C: 15.00\,/\underline{-36.87^o}kVA$$

Converting these to impedances using the formula $S = \dfrac{|V|^2}{Z}$ with $V = 120V$ yields:

$$Z_{AN} = 0.8233\,/\underline{30.96^o}\Omega = R_A + jX_A = 0.7060 + j0.4236\Omega$$
$$Z_{BN} = 0.8878\,/\underline{33.39^o}\Omega = R_B + jX_B = 0.7387 + j0.4925\Omega$$
$$Z_{CN} = 0.9600\,/\underline{36.87^o}\Omega = R_C + jX_C = 0.7680 + j0.5760\Omega$$

For the rest of this analysis we will assume that each phase of the system has an equivalent source and line impedance of $R_s + jX_s = 0.0300 + j0.0200\Omega$ and that the ground return wire has an impedance of $R_n + jZ_n = 0.0100 + j0.0080\Omega$. This yields the equivalent circuit of Figure 3.

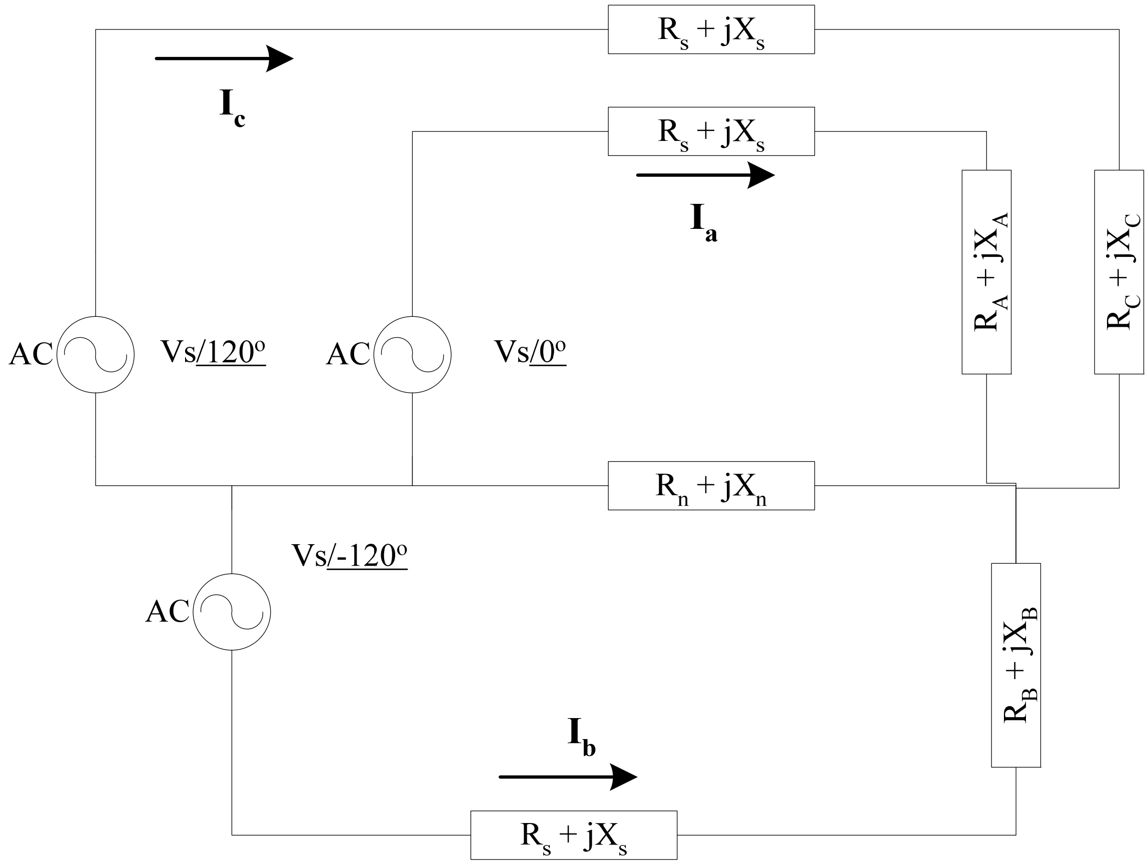

Figure 3 Equivalent Circuit Model for the Working Problem

The circuit can be analyzed using three loop equations using the currents $I_a, I_b,$ and I_c shown in Figure 3. For loop A this yields the complex equation:

Loop A:

$$-V_s\underline{0^\circ} + I_a\left(R_s + jX_s + R_A + jX_A\right) + \left(I_a + I_b + I_c\right)\left(R_n + jX_n\right) = 0$$

with loops B and C yielding similar results. Assuming that our simultaneous equation solver is not capable of handling complex numbers we can turn the loop A equation into two separate non-complex equations addressing both the real and imaginary parts. Using $I_a = I_{ar} + jI_{ai}$ and collecting terms yields:

Real A:

$$I_{ar}\left(R_s + R_A + R_n\right) - I_{ai}\left(X_s + X_A + X_n\right) + I_{br}R_n - I_{bi}X_n + I_{cr}R_n - I_{ci}X_n = 120 \quad (1)$$

Imaginary A:

$$I_{ar}\left(X_s + X_A + X_n\right) + I_{ai}\left(R_s + R_A + R_n\right) + I_{br}X_n + I_{bi}R_n + I_{cr}X_n + I_{ci}R_n = 0 \quad (2)$$

Applying the same analysis to the B and C loops yields the remaining equations for the system.

Real B:

$$I_{ar}R_n - I_{ai}X_n + I_{br}\left(R_s + R_B + R_n\right) - I_{bi}\left(X_s + X_B + X_n\right) + I_{cr}R_n - I_{ci}X_n = -60 \quad (3)$$

Imaginary B:

$$I_{ar}X_n + I_{ai}R_n + I_{br}\left(X_s + X_B + X_n\right) + I_{bi}\left(R_s + R_B + R_n\right) + I_{cr}X_n + I_{ci}R_n = -103.9 \quad (4)$$

Real C:

$$I_{ar}R_n - I_{ai}X_n + I_{br}R_n - I_{bi}X_n + I_{cr}(R_s + R_C + R_n) - I_{ci}(X_s + X_C + X_n) = -60 \quad (5)$$

Imaginary C:

$$I_{ar}X_n + I_{ai}R_n + I_{br}X_n + I_{bi}R_n + I_{cr}(X_s + X_C + X_n) + I_{ci}(R_s + R_C + R_n) = 103.9 \quad (6)$$

This yields a system of six linear equations and six unknowns $(I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, \text{and } I_{ci})$ that can be solved by any conventional means. This is shown in matrix form in Figure 4.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120.0 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix} \quad (7)$$

Figure 4 Complete System of Equations

Once the currents are known it is a simple procedure to determine the voltages across the three motor terminals $(V_{AN}, V_{BN}, \text{and } V_{CN})$ using Ohm's Law.

$$V_{AN} = (I_{ar} + jI_{ai})(R_A + jX_A)$$
$$V_{BN} = (I_{br} + jI_{bi})(R_B + jX_B)$$
$$V_{CN} = (I_{cr} + jI_{ci})(R_C + jX_C)$$

To better evaluate the imbalance, the percentage difference in currents through the actual load elements is more often considered. The best way to think of this as three people pulling and pushing together. If they do not pull and push in balance, things can become unstable. In the case of a three-phase motor, this can result in significant wobble with corresponding wear in the bearings and other parts. To determine the current through each load is again computed using Ohm's Law.

$$I_{Aload} = \frac{V_{AN}}{Z_{3\phi}}, I_{Bload} = \frac{V_{BN}}{Z_{3\phi}} I_{Cload} = \frac{V_{CN}}{Z_{3\phi}}$$

$$Z_{3\phi} = 0.7680 + j0.5760\Omega$$

where $Z_{3\phi}$ was computed earlier as Z_C. Do not forget that the lighting loads are separate.

QUESTIONS

1. What would be the ramifications of solving the problem directly using the three complex linear equations? Could we do it using an approach like Gauss-Jordan Elimination? What about some of the other numerical methods used to solve simultaneous linear equations?

2. This problem is only interesting if the ground return leg Z_{nN} is non-zero. Otherwise, we have three loop equations that are completely independent of each other and can be solved directly. Why is that the case?

3. A much more interesting and practical problem occurs when the motor load is a Delta configuration. Since it does not have the ground return line in the middle it results in additional loop equations. Sketch the equivalent circuit for a system with a Wye source and a mix of Delta and Wye loads. Write the set of equations that result from this system. Solve them.

SIMULTANEOUS LINEAR EQUATIONS	
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